Mathematical Tables
and other
Aids to Computation

A Quarterly Journal

Edited by

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Numerical Integration Constants

Tables of the b_j and x_j in the approximate quadrature formula

$$\int_0^1 x^n g(x)dx \doteq \sum_{j=1}^{m} b_j g(x_j) \tag{1}$$

are given for $n = 0(1)5$, $m = 1(1)8$ to 12D. The x_j are the zeros of certain orthonormal polynomials $q_m(x)$; the $q_m(x)$ are given explicitly for the same range of n, m. It is well known that there is equality in (1) for polynomials not exceeding degree $2m - 1$. These tables extend those given by Hammer, Marlowe, and Stroud [2].

P. C. Hammer points out that due to the relation,

$$\int_c^d \left\{ \left(\int_c^t \right)^n f(t)(dt)^n \right\} dt = \frac{(d-c)^{n+1}}{n!} \int_0^1 x^n f(d - [d-c]x)dx,$$

(see [4]), the tables given below can be applied to the evaluation of repeated integrals.

$n = 0$

m	
0	$q_0(x) = \sqrt{1}$
1	$q_1(x) = \sqrt{3}(1 - 2x)$
2	$q_2(x) = \sqrt{5}(1 - 6x + 6x^2)$
3	$q_3(x) = \sqrt{7}(1 - 12x + 30x^2 - 20x^3)$
4	$q_4(x) = \sqrt{9}(1 - 20x + 90x^2 - 140x^3 + 70x^4)$
5	$q_5(x) = \sqrt{11}(1 - 30x + 210x^2 - 560x^3 + 630x^4 - 252x^5)$
6	$q_6(x) = \sqrt{13}(1 - 42x + 420x^2 - 1680x^3 + 3150x^4 - 2772x^5 + 924x^6)$
7	$q_7(x) = \sqrt{15}(1 - 56x + 756x^2 - 4200x^3 + 11550x^4 - 16632x^5 + 12012x^6 - 3432x^7)$
8	$q_8(x) = \sqrt{17}(1 - 72x + 1260x^2 - 9240x^3 + 34650x^4 - 72072x^5 + 84084x^6 - 51480x^7 + 12870x^8)$.

$n = 1$

0	$q_0(x) = \sqrt{2}$
1	$q_1(x) = \sqrt{4}(2 - 3x)$
2	$q_2(x) = \sqrt{6}(3 - 12x + 10x^2)$
3	$q_3(x) = \sqrt{8}(4 - 30x + 60x^2 - 35x^3)$
4	$q_4(x) = \sqrt{10}(5 - 60x + 210x^2 - 280x^3 + 126x^4)$
5	$q_5(x) = \sqrt{12}(6 - 105x + 560x^2 - 1260x^3 + 1260x^4 - 462x^5)$
6	$q_6(x) = \sqrt{14}(7 - 168x + 1260x^2 - 4200x^3 + 6930x^4 - 5544x^5 + 1716x^6)$
7	$q_7(x) = \sqrt{16}(8 - 252x + 2520x^2 - 11550x^3 + 27720x^4 - 36036x^5 + 24024x^6 - 6435x^7)$
8	$q_8(x) = \sqrt{18}(9 - 360x + 4620x^2 - 27720x^3 + 90090x^4 - 168168x^5 + 180180x^6 - 102960x^7 + 24310x^8)$

$n = 2$

0 $q_0(x) = \sqrt{3}$

1 $q_1(x) = \sqrt{5}(3 - 4x)$

2 $q_2(x) = \sqrt{7}(6 - 20x + 15x^2)$

3 $q_3(x) = \sqrt{9}(10 - 60x + 105x^2 - 56x^3)$

4 $q_4(x) = \sqrt{11}(15 - 140x + 420x^2 - 504x^3 + 210x^4)$

5 $q_5(x) = \sqrt{13}(21 - 280x + 1260x^2 - 2520x^3 + 2310x^4 - 792x^5)$

6 $q_6(x) = \sqrt{15}(28 - 504x + 3150x^2 - 9240x^3 + 13860x^4 - 10296x^5 + 3003x^6)$

7 $q_7(x) = \sqrt{17}(36 - 840x + 6930x^2 - 27720x^3 + 60060x^4 - 72072x^5 + 45045x^6 - 11440x^7)$

8 $q_8(x) = \sqrt{19}(45 - 1320x + 13860x^2 - 72072x^3 + 210210x^4 - 360360x^5 + 360360x^6 - 194480x^7 + 43758x^8)$

$n = 3$

0 $q_0(x) = \sqrt{4}$

1 $q_1(x) = \sqrt{6}(4 - 5x)$

2 $q_2(x) = \sqrt{8}(10 - 30x + 21x^2)$

3 $q_3(x) = \sqrt{10}(20 - 105x + 168x^2 - 84x^3)$

4 $q_4(x) = \sqrt{12}(35 - 280x + 756x^2 - 840x^3 + 330x^4)$

5 $q_5(x) = \sqrt{14}(56 - 630x + 2520x^2 - 4620x^3 + 3960x^4 - 1287x^5)$

6 $q_6(x) = \sqrt{16}(84 - 1260x + 6930x^2 - 18480x^3 + 25740x^4 - 18018x^5 + 5005x^6)$

7 $q_7(x) = \sqrt{18}(120 - 2310x + 16632x^2 - 60060x^3 + 120120x^4 - 135135x^5 + 80080x^6 - 19448x^7)$

8 $q_8(x) = \sqrt{20}(165 - 3960x + 36036x^2 - 168168x^3 + 450450x^4 - 720720x^5 + 680680x^6 - 350064x^7 + 75582x^8)$

$n = 4$

0 $q_0(x) = \sqrt{5}$

1 $q_1(x) = \sqrt{7}(5 - 6x)$

2 $q_2(x) = \sqrt{9}(15 - 42x + 28x^2)$

3 $q_3(x) = \sqrt{11}(35 - 168x + 252x^2 - 120x^3)$

4 $q_4(x) = \sqrt{13}(70 - 504x + 1260x^2 - 1320x^3 + 495x^4)$

5 $q_5(x) = \sqrt{15}(126 - 1260x + 4620x^2 - 7920x^3 + 6435x^4 - 2002x^5)$

6 $q_6(x) = \sqrt{17}(210 - 2772x + 13860x^2 - 34320x^3 + 45045x^4 - 30030x^5 + 8008x^6)$

7 $q_7(x) = \sqrt{19}(330 - 5544x + 36036x^2 - 120120x^3 + 225225x^4 - 240240x^5 + 136136x^6 - 31824x^7)$

8 $q_8(x) = \sqrt{21}(495 - 10296x + 84084x^2 - 360360x^3 + 900900x^4 - 1361360x^5 + 1225224x^6 - 604656x^7 + 125970x^8)$

$n = 5$

0 $q_0(x) = \sqrt{6}$

1 $q_1(x) = \sqrt{8}(6 - 7x)$

2 $q_2(x) = \sqrt{10}(21 - 56x + 36x^2)$

3 $q_3(x) = \sqrt{12}(56 - 252x + 360x^2 - 165x^3)$

4 $q_4(x) = \sqrt{14}(126 - 840x + 1980x^2 - 1980x^3 + 715x^4)$

5 $q_5(x) = \sqrt{16}(252 - 2310x + 7920x^2 - 12870x^3 + 10010x^4 - 3003x^5)$

6 $q_6(x) = \sqrt{18}(462 - 5544x + 25740x^2 - 60060x^3 + 75075x^4 - 48048x^5 + 12376x^6)$

7 $q_7(x) = \sqrt{20}(792 - 12012x + 72072x^2 - 225225x^3 + 400400x^4 - 408408x^5 + 222768x^6 - 50388x^7)$

8 $q_8(x) = \sqrt{22}(1287 - 24024x + 180180x^2 - 720720x^3 + 1701700x^4 - 2450448x^5 + 2116296x^6 - 1007760x^7 + 203490x^8)$

$n = 0$

m	j	x_j	b_j
1	1	.50000 00000 00	1.00000 00000 00
2	1	.21132 48654 05	.50000 00000 00
	2	.78867 51345 95	.50000 00000 00
3	1	.11270 16653 79	.27777 77777 78
	2	.50000 00000 00	.44444 44444 44
	3	.88729 83346 21	.27777 77777 78
4	1	.06943 18442 03	.17392 74225 69
	2	.33000 94782 08	.32607 25774 31
	3	.66999 05217 92	.32607 25774 31
	4	.93056 81557 97	.17392 74225 69
5	1	.04691 00770 31	.11846 34425 28
	2	.23076 53449 47	.23931 43352 50
	3	.50000 00000 00	.28444 44444 44
	4	.76923 46550 53	.23931 43352 50
	5	.95308 99229 69	.11846 34425 28
6	1	.03376 52428 98	.08566 22461 90
	2	.16939 53067 67	.18038 07865 24
	3	.38069 04069 58	.23395 69672 86
	4	.61930 95930 42	.23395 69672 86
	5	.83060 46932 33	.18038 07865 24
	6	.96623 47571 02	.08566 22461 90
7	1	.02544 60438 29	.06474 24830 84
	2	.12923 44072 00	.13985 26957 45
	3	.29707 74243 11	.19091 50252 53
	4	.50000 00000 00	.20897 95918 37
	5	.70292 25756 89	.19091 50252 53
	6	.87076 55928 00	.13985 26957 45
	7	.97455 39561 71	.06474 24830 84
8	1	.01985 50717 51	.05061 42681 45
	2	.10166 67612 93	.11119 05172 27
	3	.23723 37950 42	.15685 33229 39
	4	.40828 26787 52	.18134 18916 89
	5	.59171 73212 48	.18134 18916 89
	6	.76276 62049 58	.15685 33229 39
	7	.89833 32387 07	.11119 05172 27
	8	.98014 49282 49	.05061 42681 45

$n = 1$

m	j	x_j	b_j
1	1	.66666 66666 67	.50000 00000 00
2	1	.35505 10257 22	.18195 86182 56
	2	.84494 89742 78	.31804 13817 44
3	1	.21234 05382 39	.06982 69799 01
	2	.59053 31355 59	.22924 11063 60
	3	.91141 20404 87	.20093 19137 39
4	1	.13975 98643 44	.03118 09709 50
	2	.41640 95676 31	.12984 75476 08
	3	.72315 69863 62	.20346 45680 10
	4	.94289 58038 85	.13550 69134 31
5	1	.09853 50857 99	.01574 79145 22
	2	.30453 57266 46	.07390 88700 73
	3	.56202 51897 53	.14638 69870 85
	4	.80198 65821 26	.16717 46380 94
	5	.96019 01429 49	.09678 15902 27
6	1	.07305 43286 80	.00873 83018 14
	2	.23076 61379 70	.04395 51655 51
	3	.44132 84812 28	.09866 11508 91
	4	.66301 53097 19	.14079 25537 88
	5	.85192 14003 32	.13554 24972 32
	6	.97068 35728 40	.07231 03307 26
7	1	.05626 25605 37	.00521 43622 03
	2	.18024 06917 37	.02740 83567 22
	3	.35262 47171 13	.06638 46964 65
	4	.54715 36263 31	.10712 50656 96
	5	.73421 01772 15	.12739 08973 00
	6	.88532 09468 39	.11050 92581 91
	7	.97752 06135 61	.05596 73634 23
8	1	.04463 39552 90	.00329 51914 42
	2	.14436 62570 42	.01784 29026 56
	3	.28682 47571 44	.04543 93195 05
	4	.45481 33151 97	.07919 95994 92
	5	.62806 78354 17	.10604 73594 36
	6	.78569 15206 04	.11250 57994 71
	7	.90867 63921 00	.09111 90236 36
	8	.98222 00848 53	.04455 08043 62

$n = 2$

1	1	.75000 00000 00	.33333 33333 33
2	1	.45584 81559 89	.10078 58820 80
	2	.87748 51773 45	.23254 74512 54
3	1	.29499 77901 12	.02995 07030 09
	2	.65299 62339 62	.14624 62692 60
	3	.92700 59759 27	.15713 63610 65

m	j	x_j	b_j
4	1	.20414 85821 03	.01035 22407 50
	2	.48295 27048 96	.06863 38871 73
	3	.76139 92624 48	.14345 87897 99
	4	.95149 94505 53	.11088 84156 11
5	1	.14894 57870 53	.00411 38252 03
	2	.36566 65273 69	.03205 56007 23
	3	.61011 36129 34	.08920 01612 22
	4	.82651 96792 28	.12619 89619 00
	5	.96542 10600 82	.08176 47842 86
6	1	.11319 43838 22	.00183 10758 07
	2	.28431 88726 88	.01572 02971 85
	3	.49096 35868 35	.05128 95711 30
	4	.69756 30819 77	.09457 71867 49
	5	.86843 60583 42	.10737 64997 37
	6	.97409 54449 06	.06253 87027 27
7	1	.08881 68334 37	.00089 26880 34
	2	.22648 27534 09	.00816 29256 32
	3	.39997 84867 21	.02942 22112 90
	4	.58599 78554 03	.06314 63787 09
	5	.75944 58739 52	.09173 38032 80
	6	.89691 09708 52	.09069 88246 13
	7	.97986 72262 27	.04927 65017 76
8	1	.07149 10350 40	.00046 85177 84
	2	.18422 82964 17	.00447 45217 13
	3	.33044 77281 76	.01724 68637 80
	4	.49440 29218 16	.04081 44263 89
	5	.65834 80085 23	.06844 71834 22
	6	.80452 48315 11	.08528 47691 72
	7	.91709 93825 14	.07681 80932 67
	8	.98390 22404 48	.03977 89578 07
		$n = 3$	
1	1	.80000 00000 00	.25000 00000 00
2	1	.52985 79358 95	.06690 52498 07
	2	.89871 34926 77	.18309 47501 93
3	1	.36326 46302 17	.01647 90592 83
	2	.69881 12691 64	.10459 98975 57
	3	.93792 41006 20	.12892 10431 61
4	1	.26147 77888 31	.00465 83670 60
	2	.53584 64460 88	.04254 17241 43
	3	.79028 32299 69	.10900 43689 39
	4	.95784 70805 66	.09379 55398 59
5	1	.19621 20073 97	.00152 06893 71
	2	.41710 02118 22	.01695 73248 63
	3	.64857 00042 37	.06044 49532 04
	4	.84560 51499 74	.10031 65044 65
	5	.96943 57034 93	.07076 05280 97

m	j	x_j	b_j
6	1	.15227 31617 84	.00056 17108 74
	2	.33130 04570 38	.00708 53159 32
	3	.53241 15667 29	.03052 61922 26
	4	.72560 27783 30	.06844 32817 68
	5	.88161 66844 37	.08830 09912 41
	6	.97679 53516 82	.05508 25079 60
7	1	.12142 71288 32	.00022 99041 27
	2	.26836 34403 11	.00314 75964 07
	3	.44086 64606 23	.01531 21671 26
	4	.61860 40284 32	.04099 51686 04
	5	.78025 35519 66	.06975 00981 07
	6	.90636 25341 45	.07655 65613 63
	7	.98176 99145 14	.04400 85042 65
8	1	.09900 17577 10	.00010 24600 78
	2	.22124 35073 50	.00148 56840 83
	3	.36912 39000 12	.00785 50738 41
	4	.52854 54312 02	.02363 15806 68
	5	.68399 32484 32	.04745 43798 12
	6	.82028 39496 79	.06736 18393 93
	7	.92409 37128 99	.06618 20353 22
	8	.98529 34400 85	.03592 69468 03

$n = 4$

1	1	.83333 33333 33	.20000 00000 00
2	1	.58633 65823 23	.04908 24922 78
	2	.91366 34176 77	.15091 75077 22
3	1	.42011 30593 37	.01046 90421 83
	2	.73388 93552 08	.08027 66734 56
	3	.94599 75854 55	.10925 42843 61
4	1	.31213 54928 47	.00251 63516 47
	2	.57891 56595 62	.02916 93821 62
	3	.81289 15166 16	.08706 77120 64
	4	.96272 39976 41	.08124 65541 27
5	1	.23979 20448 02	.00069 69770 78
	2	.46093 36745 32	.01021 05417 25
	3	.68005 92327 41	.04402 44695 05
	4	.86088 63436 76	.08271 27131 02
	5	.97261 44185 34	.06235 52985 89
6	1	.18946 95839 22	.00021 94139 95
	2	.37275 11560 14	.00372 67844 39
	3	.56757 23728 55	.01995 62646 93
	4	.74883 64975 06	.05223 99542 93
	5	.89238 51584 47	.07464 91503 13
	6	.97898 52312 56	.04920 84322 67

m	j	x_j	b_j
7	1	.15324 14388 69	.00007 70737 07
	2	.30632 65225 42	.00144 70087 89
	3	.47654 00930 01	.00892 69676 13
	4	.64638 93025 20	.02854 78427 53
	5	.79771 66898 15	.05522 48741 64
	6	.91421 99005 65	.06602 18459 35
	7	.98334 38304 67	.03975 43870 40
8	1	.12637 29743 78	.00002 97092 30
	2	.25552 90520 78	.00059 89500 38
	3	.40364 12988 94	.00407 79241 35
	4	.55831 66757 90	.01490 99334 50
	5	.70600 95428 83	.03471 99506 61
	6	.83367 15420 39	.05491 00972 91
	7	.92999 57160 53	.05800 05652 68
	8	.98646 31978 85	.03275 28699 27

$n = 5$

1	1	.85714 28571 43	.16666 66666 67
2	1	.63079 15938 30	.03833 75627 37
	2	.92476 39617 26	.12832 91039 30
3	1	.46798 32354 55	.00729 70036 38
	2	.76162 39696 99	.06459 66122 96
	3	.95221 09766 64	.09477 30507 33
4	1	.35689 37290 50	.00153 44797 48
	2	.61466 93898 55	.02142 84046 31
	3	.83107 90038 60	.07205 63641 65
	4	.96658 86464 65	.07164 74181 22
5	1	.27969 31248 12	.00036 97154 99
	2	.49870 98270 30	.00672 96904 30
	3	.70633 38189 21	.03376 77449 58
	4	.87340 27278 72	.07007 13397 05
	5	.97519 38346 98	.05572 81760 75
6	1	.22446 89954 04	.00010 13258 20
	2	.40953 33504 56	.00218 79256 55
	3	.59778 90484 17	.01396 96530 85
	4	.76841 36046 08	.04148 63469 81
	5	.90135 07338 49	.06445 88591 64
	6	.98079 72084 41	.04446 25559 61
7	1	.18382 87683 01	.00003 11046 06
	2	.34080 75951 06	.00075 53838 32
	3	.50794 05240 29	.00566 04137 04
	4	.67036 34101 11	.02095 92981 75
	5	.81258 84659 90	.04510 49815 82
	6	.92085 64172 55	.05790 76135 39
	7	.98466 74507 86	.03624 78712 29

m	j	x_j	b_j
8	1	.15315 06616 10	.00001 05316 45
	2	.28726 44038 84	.00027 83586 15
	3	.43462 74066 99	.00233 53415 00
	4	.58451 85665 63	.01004 46143 88
	5	.72512 64097 13	.02648 53011 18
	6	.84518 94879 31	.04588 56532 01
	7	.93504 35074 56	.05153 42238 32
	8	.98746 05085 24	.03009 26423 68

The polynomials $q_i(x)$ are obtained by orthonormalization of the sequence $1, x, x^2, \cdots$ with weight function x^n in the interval $(0, 1)$. The Jacobi polynomials are defined by Courant and Hilbert [1] as follows:

$$J_{m,n} = \frac{n!}{(m+n)!} x^{-n} \frac{d^m}{dx^m} x^{m+n}(1-x)^m.$$

(Editor's note. The author's $J_{m,n}(x)$ is Courant's $G_m(n+1, n+1, x)$, and this, except for a normalizing factor, is $P_m^{(n,0)}(2x-1)$ of the Bateman Project [3].) Since [1] we have

$$\int_0^1 x^n J_{k,n} J_{m,n} dx = \frac{1}{(n+2m+1)\binom{m+n}{n}^2} \delta_{k,m},$$

the corresponding orthonormal sequence is

$$q_m(x) = \sqrt{n+2m+1}\binom{m+n}{n} J_{m,n}(x)$$

$$= \sqrt{n+2m+1} \sum_{k=0}^{m} (-1)^k \binom{m+n+k}{m}\binom{m}{k} x^k.$$

The explicit formula for b_j is

$$b_j = \{\sum_{i=0}^{m-1} [q_i(x_j)]^2\}^{-1}.$$

Within the range of values given in tables I, II, III of the Hammer, Marlowe, Stroud paper, it is seen that these polynomials are identical to those given therein except for a factor of -1 in the ones of odd degree associated with weight function x^2. Obviously, this has no effect on the values in the table.

The Newton-Raphson method with synthetic division was used to find the zeros. With the polynomials of higher degree (*i.e.*, for $m = 6$, 7, and 8) and especially for the higher values of n where the coefficients are large, the round-off errors in the zeros for 18 decimal places were seen to be considerable. As a check these preliminary values were substituted back into the original equation and the process was carried out again with the zeros recorded after each iteration.

Synthetic division was not used this time because it was felt that the round-off error would be carried into the reduced polynomial. Observing the erratic property of the last digits it was concluded that the limit of accuracy in the worst case is about 12 decimal places. There seems to be no question for values of m from 1 to 5 where $n = 0$, 1, and 2.

A check of the b_j was made through the fact that

$$\sum_{j=1}^{m} b_j = \int_0^1 x^n dx = \frac{1}{n+1}$$

for each polynomial of degree m associated with x^n. Here again the limit of accuracy is put at about the twelfth decimal place for the worst possible cases (*i.e.*, with m and n both large). If one wishes to round off values to ten or eleven decimal places there should be no doubt of the accuracy.

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Tables of the Exponential Integral Ei(x)

In some molecular structure calculations it is desirable to have values of the integral Ei(x) to higher accuracy than is provided by the standard tables [1]. Direct computation of the values needed is extremely tedious over a wide range of the argument intermediate between the regions where expansions about zero and infinity are useful. However, if a net of sufficiently accurate values of Ei(x) has already been computed, then the generation of additional values intermediate between these becomes a problem of a considerably lesser magnitude. A few such values have been published [2, 3]. Kotani [4] has given a considerable number of values in his most valuable compilation, but unfortunately they are of somewhat fewer significant figures than under discussion here. A prohibitive amount of labor would be required to make such an extensive accurate table that only simple interpolation formulas need be used. However, a most useful intermediate objective could be achieved if a table were prepared with entries sufficiently closely

spaced that rapidly convergent series expansions could be used to obtain further values. Accordingly, an 18-significant digit table was prepared by the means described below.

The two usual ways of obtaining values of Ei(x) *ab initio* are from the Taylor series

$$\text{Ei}(x) = \gamma + \ln|x| + x + \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!} + \cdots \tag{1}$$

and from the asymptotic formula

$$\text{Ei}(x) \sim e^x\left[\frac{1}{x} + \frac{1}{x^2} + \frac{2!}{x^3} + \frac{3!}{x^4} + \cdots\right]. \tag{2}$$

The Taylor series becomes inconvenient beyond about $|x| = 4$, and the asymptotic formula is incapable of giving sufficiently accurate values for $|x| < 50$. It is necessary to generate values in the range $4 < |x| < 50$ with the aid of a suitable recursion formula.

Two recursion formulas have been employed recently for this purpose. The first, used by Ruedenberg, Roothaan, and Jaunzemis [5], may be written in the form

$$\text{Ei}(x) = \text{Ei}(x - h) + \sum_{k=1}^{\infty} \frac{(-h)^k}{k!} A_{k-1}(-x) \tag{3}$$

where $A_n(x)$ is defined as

$$A_n(x) = x^{-n-1}e^{-x}n!\sum_{k=0}^{n}\frac{x^k}{k!}. \tag{4}$$

The integrals $A_n(x)$ have been tabulated by Kotani [4] for positive x, though not all to sufficient accuracy for use in (3). $A_n(x)$ for negative x can be obtained, also not with sufficient accuracy for use in (3) from Kotani's tables of $B_n(x)$ and the relation $B_n(x) = (-1)^{n+1}A_n(-x) - A_n(x)$.

A second recursion formula for Ei(x), used by Kotani [4], may be cast in the form

$$e^{-x}\,\text{Ei}(x) = e^{-h}[e^{-(x-h)}\,\text{Ei}(x - h)] + x^{-1}\sum_{k=0}^{\infty} x^{-k}R_k(h) \tag{5}$$

where the coefficients $R_n(h)$ are related to the A_n by

$$R_n(h) = n! - h^{n+1}A_n(h). \tag{6}$$

This second expansion is far more convenient than the first, as very little computation is required to apply it for given h to different values of x. Moreover, the coefficients $R_n(h)$ may be obtained without great labor by consideration of the recursion relation between $R_n(h)$ and $R_{n-1}(h)$,

$$R_n(h) = nR_{n-1}(h) - h^n e^{-h}, \quad n \geq 1, \tag{7}$$

together with the value $R_0(h) = 1 - e^{-h}$. Equation (7) may be verified by induction. A third recursion formula, intermediate in character between (3) and (5), was given by Gram [3].

Equations (5) and (7) were employed to compute the values of $e^{-x}\,\mathrm{Ei}(x)$ presented here for $|x| \geq 4$. By using the formula to proceed from smaller to larger x, no loss in significant figures accompanies repeated use of the formula, and the final accuracy is determined by the number of figures carried in the R_n. Starting values of Ei(−50) and Ei(4) were computed by (2) and (1), respectively, and the values of Ei(−4) and Ei(50) obtained by repeated use of the recursion formulas were checked against values computed, respectively, from (1) and (2). Twenty-one significant figures were carried throughout the calculations, and the final results were rounded off to 18. Some intermediate values were also obtained in more than one way as a check. The same work sheets were used to compute for both positive and negative values of x, thus further reducing the possibility of error in formulation of the $x^{-k}R_k$. All check conditions were satisfied to 20 or more significant digits.

The 21-figure values of $e^{-x}\,\mathrm{Ei}(x)$ were multiplied by 19-figure values of e^x [6] to obtain the values of Ei(x) presented to 18 figures. Table 1 gives our final values of Ei($-x$) and $-e^x\,\mathrm{Ei}(-x)$, together with additional values obtained by Bretschneider [2]. In table 2 are presented the corresponding quantities for positive x. Table 3 gives values of the $R_n(h)$ to sufficient precision to interpolate from the nearest table value anywhere for $|x| > 4$. Miscellaneous constants used are listed in table 4.

TABLE 1. *Exponential Integrals, Negative Arguments*

The numbers in parentheses are the powers of 10 by which the entries so marked must be multiplied.

x	$-\mathrm{Ei}(-x)$	$-e^x\,\mathrm{Ei}(-x)$
1.0	0.21938 39343 95520 274 (0)	0.59634 73623 23194 0743
2.0	0.48900 51070 80611 196 (− 1)	0.36132 86168 88222 5846
3.0	0.13048 38109 41970 374 (− 1)	0.26208 37402 55318 4961
4.0	0.37793 52409 84890 648 (− 2)	0.20634 56499 01055 8331
4.4	0.23360 10040 65582 899 (− 2)	0.19027 00470 21504 7599
4.8	0.14529 93936 87832 422 (− 2)	0.17655 38999 22275 4908
5.2	0.90862 16124 48659 585 (− 3)	0.16470 78767 04783 2961
5.6	0.57084 01696 48211 674 (− 3)	0.15437 02562 92501 3756
6.0	0.36008 24521 62658 659 (− 3)	0.14526 76292 33886 8938
6.4	0.22794 79559 29372 847 (− 3)	0.13718 93461 69177 0926
6.8	0.14475 77922 44899 578 (− 3)	0.12997 03917 12376 9249
7.2	0.92188 11688 71620 423 (− 4)	0.12347 95998 70253 3931
7.6	0.58858 77207 53838 951 (− 4)	0.11761 13567 51900 9850
8.0	0.37665 62284 39249 018 (− 4)	0.11227 96392 53499 3118
9.0	0.12447 35417 80062 721 (− 4)	0.10086 19555 80640 9291
10.0	0.41569 68929 68532 428 (− 5)	0.09156 33339 39788 0819

TABLE 1—*Continued*

x	$-\mathrm{Ei}(-x)$	$-e^x\,\mathrm{Ei}(-x)$
11.0	0.14003 00304 24744 178 (− 5)	0.08384 17788 60345 9505
12.0	0.47510 81824 67249 393 (− 6)	0.07732 61331 38919 2306
13.0	0.16218 66218 80143 287 (− 6)	0.07175 35335 24462 3954
14.0	0.55656 31111 14518 212 (− 7)	0.06693 25181 83439 6292
15.0	0.19186 27892 14786 698 (− 7)	0.06272 02791 07409 2418
16.0	0.66404 87249 44104 278 (− 8)	0.05900 81036 08556 4362
17.0	0.23064 31989 82165 449 (− 8)	0.05571 17557 43476 7274
18.0	0.80360 90344 82867 766 (− 9)	0.05276 49444 02625 0680
19.0	0.28078 29097 06079 527 (− 9)	0.05011 47797 95479 9196
20.0	0.98355 25290 64988 169 (−10)	0.04771 85454 95960 8417
21.0	0.34532 01267 14675 627 (−10)	0.04554 13616 54505 1706
22.0	0.12149 37895 62043 727 (−10)	0.04355 44646 92617 8827
23.0	0.42826 84795 66567 262 (−11)	0.04173 39215 56000 6238
24.0	0.15123 05893 99970 577 (−11)	0.04005 96555 23840 3225
25.0	0.53488 99755 34021 664 (−12)	0.03851 46988 44904 0221
26.0	0.18946 85885 67497 824 (−12)	0.03708 46128 39349 6946
27.0	0.67206 37435 26204 038 (−13)	0.03575 70332 31537 5283
28.0	0.23869 41511 93373 316 (−13)	0.03452 13102 36847 3608
29.0	0.84877 59778 35356 284 (−14)	0.03336 82211 22561 0743
30.0	0.30215 52010 68881 254 (−14)	0.03228 97387 58980 1252
31.0	0.10767 67038 61623 826 (−14)	0.03127 88438 29256 7742
32.0	0.38409 61801 22506 683 (−15)	0.03032 93713 77333 2475
33.0	0.13713 84348 44874 657 (−15)	0.02943 58845 81341 5053
34.0	0.49006 76118 39278 771 (−16)	0.02859 35702 76001 3970
35.0	0.17527 05938 99473 720 (−16)	0.02779 81519 71952 9646
36.0	0.62733 39009 76224 159 (−17)	0.02704 58170 44635 3756
37.0	0.22470 20697 58857 122 (−17)	0.02633 31554 69674 7075
38.0	0.80541 06914 29074 987 (−18)	0.02565 71080 22614 4181
39.0	0.28887 79301 52270 100 (−18)	0.02501 49222 79330 0217
40.0	0.10367 73261 45165 697 (−18)	0.02440 41150 79628 5763
41.0	0.37231 66776 45997 772 (−19)	0.02382 24403 72451 8529
42.0	0.13377 90881 00117 751 (−19)	0.02326 78615 63199 8260
43.0	0.48094 96556 95001 785 (−20)	0.02273 85276 44261 4456
44.0	0.17299 59874 28164 776 (−20)	0.02223 27525 18155 0389
45.0	0.62256 90809 46238 364 (−21)	0.02174 89970 25785 2704
46.0	0.22415 31759 74429 975 (−21)	0.02128 58532 75605 0273
47.0	0.80741 97842 72581 395 (−22)	0.02084 20309 37076 1867
48.0	0.29096 64190 40584 234 (−22)	0.02041 63452 16965 1554
49.0	0.10489 81164 23680 237 (−22)	0.02000 77062 82189 8425
50.0	0.37832 64029 55045 902 (−23)	0.01961 51099 30114 8704

TABLE 2. *Exponential Integrals, Positive Arguments*

The numbers in parentheses are the powers of 10 by which the entries so marked must be multiplied.

x	Ei(x)	e^{-x} Ei(x)
1.0	0.18951 17816 35593 676 (1)	0.69717 48832 35066 0688
2.0	0.49542 34356 00189 016 (1)	0.67048 27097 90073 2810
3.0	0.99338 32570 62541 656 (1)	0.49457 64013 48641 2350
4.0	0.19630 87447 00562 200 (2)	0.35955 20078 63620 6962
4.4	0.26008 97327 16051 460 (2)	0.31932 10053 85318 4003
4.8	0.34697 88987 37753 276 (2)	0.28555 48567 95924 4435
5.2	0.46624 85050 57967 478 (2)	0.25720 89914 23568 2427
5.6	0.63101 78597 42992 615 (2)	0.23334 18047 99632 7448
6.0	0.85989 76214 24392 048 (2)	0.21314 73100 81593 6031
6.4	0.11793 48657 00181 887 (3)	0.19595 55338 64928 3941
6.8	0.16270 70875 71431 415 (3)	0.18121 91105 15268 3455
7.2	0.22568 78077 01063 810 (3)	0.16849 53143 53259 2881
7.6	0.31457 18784 98083 578 (3)	0.15742 79475 14471 5487
8.0	0.44037 98995 34838 269 (3)	0.14773 09983 73400 9966
9.0	0.10378 78290 71708 959 (4)	0.12808 43565 23213 8681
10.0	0.24922 28976 24187 776 (4)	0.11314 70204 73410 7780
11.0	0.60714 06374 09861 151 (4)	0.10140 28126 36185 3129
12.0	0.14959 53266 63975 289 (5)	0.09191 45454 08896 5894
13.0	0.37197 68849 06890 356 (5)	0.08407 90291 67225 1376
14.0	0.93192 51363 39653 713 (5)	0.07749 22514 92093 0115
15.0	0.23495 58524 90768 304 (6)	0.07187 35404 92410 7085
16.0	0.59556 09986 70837 002 (6)	0.06702 15610 41399 0873
17.0	0.15166 37894 04251 688 (7)	0.06278 78642 32855 1665
18.0	0.38779 04330 59744 350 (7)	0.05906 04044 06932 4489
19.0	0.99509 07251 04684 476 (7)	0.05575 29076 96429 0258
20.0	0.25615 65266 40565 888 (8)	0.05279 77952 79648 1322
21.0	0.66127 18635 54849 213 (8)	0.05014 13386 46825 6145
22.0	0.17114 46713 00363 668 (9)	0.04774 02599 85690 4452
23.0	0.44396 63698 30271 221 (9)	0.04555 92944 77218 8886
24.0	0.11541 15391 84918 295 (10)	0.04356 94088 38540 5760
25.0	0.30059 50906 52554 869 (10)	0.04174 64774 50664 5301
26.0	0.78429 40991 89818 637 (10)	0.04007 02837 69455 1129
27.0	0.20496 49711 98808 124 (11)	0.03852 37569 74926 3125
28.0	0.53645 11859 23146 942 (11)	0.03709 23813 73152 0594
29.0	0.14059 91957 58406 905 (12)	0.03576 37344 29948 2813
30.0	0.36897 32094 07274 197 (12)	0.03452 71217 92361 8461
31.0	0.96945 55759 68393 966 (12)	0.03337 32862 79497 7246
32.0	0.25500 43566 35778 693 (13)	0.03229 41738 81757 5328
33.0	0.67146 40184 07649 756 (13)	0.03128 27441 22948 5202
34.0	0.17698 03724 41162 685 (14)	0.03033 28152 54435 4435
35.0	0.46690 55014 46615 954 (14)	0.02943 89370 25758 9265

TABLE 2—*Continued*

x	Ei(x)	e^{-x} Ei(x)
36.0	0.12328 52079 91209 769 (15)	0.02859 62854 56693 8000
37.0	0.32579 88998 67226 400 (15)	0.02780 05752 89820 6486
38.0	0.86163 88199 96578 654 (15)	0.02704 79867 47528 2957
39.0	0.22804 46200 30190 260 (16)	0.02633 51039 35588 4305
40.0	0.60397 18263 61124 158 (16)	0.02565 88627 85975 1452
41.0	0.16006 64914 32450 411 (17)	0.02501 65068 56911 8321
42.0	0.42447 96092 13685 076 (17)	0.02440 55496 39127 1203
43.0	0.11263 48290 16696 676 (18)	0.02382 37422 76697 9580
44.0	0.29904 44718 63233 668 (18)	0.02326 90458 15478 3594
45.0	0.79439 16035 70445 377 (18)	0.02273 96072 54528 2793
46.0	0.21113 42388 64782 419 (19)	0.02223 37388 05633 9048
47.0	0.56143 29680 81034 311 (19)	0.02174 98998 70127 3613
48.0	0.14936 30213 11299 314 (20)	0.02128 66813 26253 2629
49.0	0.39754 42747 90374 484 (20)	0.02084 27917 88506 7972
50.0	0.10585 63689 71316 910 (21)	0.02041 70455 55943 9873

TABLE 3. *Interpolation Coefficients*

n	$R_n(1)$	$R_n(0.5)$	$R_n(0.4)$
0	0.63212 05588 28557 67840	0.39346 93402 87366 57640	0.32967 99539 64360 69926
1	0.26424 11176 57115 35681	0.09020 40104 31049 86459	0.06155 19355 50104 97896
2	0.16060 27941 42788 3920	0.02877 53559 33941 3733	0.01585 26637 34507 6698
3	0.11392 89412 56922 854	0.01050 97353 37744 942	0.00465 75082 57242 094
4	0.08783 63238 56249 10	0.00413 07751 18940 18	0.00146 98398 50456 01
5	0.07130 21781 09803 2	0.00169 97924 78681 1	0.00048 51219 80875 1
6	0.05993 36274 87377	0.00072 17133 14077	0.00016 51009 76689
7	0.05165 59512 4019	0.00031 34724 1953	0.00005 74544 7340
8	0.04536 81687 501	0.00013 85189 668	0.00002 03348 418
9	0.04043 40775 80	0.00006 20405 06	0.00000 72931 98
10	0.03646 13346 2	0.00002 80899 6	0.00000 26438 3
11	0.03319 52397	0.00001 28320 4	0.00000 09668 6
12	0.03046 3435	0.00000 59057	0.00000 03562
13	0.02814 522	0.00000 2735	0.00000 0132
14	0.02615 36	0.00000 127	0.00000 005
15	0.02442 4	0.00000 06	
16	0.02291		
17	0.0216		
18	0.020		
19	0.02		

n	$R_n(0.3)$	$R_n(0.2)$	$R_n(0.1)$
0	0.25918 17793 18282 13393	0.18126 92469 22018 14133	0.09516 25819 64040 42684
1	0.03693 63131 13766 77411	0.01752 30963 06421 76960	0.00467 88401 60444 46952
2	0.00719 89863 66178 9403	0.00229 69624 89724 2648	0.00030 93061 40529 34331
3	0.00159 48671 40130 438	0.00034 10414 44548 9397	0.00002 30810 03552 07035
4	0.00037 88409 72999 84	0.00005 41965 73270 988	0.00000 18402 72404 6854
5	0.00009 40165 88742 6	0.00000 89890 25369 98	0.00000 01529 87843 068
6	0.00002 40430 49579	0.00000 15353 84022 91	0.00000 00130 89640 37
7	0.00000 62844 0219	0.00000 02679 34521 0	0.00000 00011 43740 8
8	0.00000 16701 340	0.00000 00475 25440 2	0.00000 00001 01552 1
9	0.00000 04496 81	0.00000 00085 38816	0.00000 00000 09131
10	0.00000 01223 6	0.00000 00015 5013	0.00000 00000 0083
11	0.00000 00335 8	0.00000 00002 8382	0.00000 00000 0008
12	0.00000 00093	0.00000 00000 523	
13	0.00000 0003	0.00000 00000 097	
14		0.00000 00000 02	

TABLE 3—*Continued*

n	$R_n(0.05)$	$R_n(0.04)$	$R_n(0.03)$
0	0.04877 05754 99285 99091	0.03921 05608 47676 79056	0.02955 44664 51491 82307
1	0.00120 91042 74250 29045	0.00077 89832 81583 86218	0.00044 11004 45036 57776
2	0.00004 01349 87248 79589	0.00002 07034 60524 00723	0.00000 87999 09879 49816
3	0.00000 15012 83683 7984	0.00000 06198 57466 2730	0.00000 01977 00232 6848
4	0.00000 00599 50832 064	0.00000 00198 08900 862	0.00000 00047 40048 565
5	0.00000 00024 94965 16	0.00000 00006 59665 74	0.00000 00001 18416 36
6	0.00000 00001 06831 23	0.00000 00000 22600 90	0.00000 00000 03043 36
7	0.00000 00000 04670 6	0.00000 00000 00790 6	0.00000 00000 00079 9
8	0.00000 00000 00207 5	0.00000 00000 00028 1	0.00000 00000 00002 1
9	0.00000 00000 00009	0.00000 00000 00001	

n	$R_n(0.02)$	$R_n(0.01)$
0	0.01980 13266 93244 69778	0.00995 01662 50831 94643
1	0.00019 73532 27109 59173	0.00004 96679 13340 26589
2	0.00000 26269 84896 4813	0.00000 03308 43305 6150
3	0.00000 00393 65302 9900	0.00000 00024 80083 0958
4	0.00000 00006 29424 231	0.00000 00000 19834 046
5	0.00000 00000 10485 40	0.00000 00000 00165 24
6	0.00000 00000 00179 69	0.00000 00000 00001 42
7	0.00000 00000 00003 1	
8	0.00000 00000 00000 1	

TABLE 4. *Constants*

$\gamma = 0.57721\ 56649\ 01532\ 86060\ 6512$ (Euler's constant)

$\log_{10} e = 0.43429\ 44819\ 03251\ 82765\ 1129$

h	e^h	e^{-h}
0.01	1.01005 01670 84168 05754 21655	0.99004 98337 49168 05357 39060
0.02	1.02020 13400 26755 81016 01439	0.98019 86733 06755 30222 08141
0.03	1.03045 45339 53516 85561 24400	0.97044 55335 48508 17693 25284
0.04	1.04081 07741 92388 22675 70448	0.96078 94391 52323 20943 92107
0.05	1.05127 10963 76024 03969 75176	0.95122 94245 00714 00909 14253
0.10	1.10517 09180 75647 62481 17078	0.90483 74180 35959 57316 42491
0.20	1.22140 27581 60169 83392 10720	0.81873 07530 77981 85866 99355
0.30	1.34985 88075 76003 10398 37443	0.74081 82206 81717 86606 68738
0.40	1.49182 46976 41270 31782 48530	0.67032 00460 35639 30074 44329
0.50	1.64872 12707 00128 14684 86508	0.60653 06597 12633 42360 37995
1.00	2.71828 18284 59045 23536 02875	0.36787 94411 71442 32159 55238

Our tables agree entirely with the values Bretschneider [2] and Gram [3] give for integers from -10 to 20. They differ from Kotani's table [4] by 1 in his last place for a number of values of the argument, and by larger amounts for the following values of x: 6.0, 8.0, 9.0, 10.0, 12.0, -18.0, -24.0, -32.0, -33.0, -43.0, -48.0.

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On the Computation of log Z and arc tan Z

In a previous note, Clenshaw [1] has given numerical values of coefficients for the expansion of some transcendental functions in Chebyshev polynomials. In particular, he tabulates the coefficients for log $(1 + x)$ and arc tan x to nine decimal places. Here, treating these functions in a more general form, we determine precise theoretical coefficients and show that the development leads to formulas for computation in the complex domain.

The Chebyshev polynomials of the first kind which we use in the range $-1 \leq x \leq 1$ are defined as

$$T_n(x) = \cos n\theta, \quad x = \cos\theta. \tag{1}$$

For a discussion of the properties of these functions, see the work of Lanczos [2]. If $f(x)$ is bounded and continuous in a given range, then the expansion

$$f(x) = \tfrac{1}{2}C_0 + \sum_{k=1}^{\infty} C_k T_k(x) \tag{2}$$

is convergent, and

$$C_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_k(x)dx}{\sqrt{1-x^2}}. \tag{3}$$

The transformation $x = 2y - 1$ shifts the range to $0 \leq y \leq 1$, and we denote the "shifted" polynomial as $B_n(y)$.

Consider $f(y) = 1/(y + a)$. Utilizing the above together with a known formula [3], we find

$$1/(y+a) = [1 + 2\sum_{k=1}^{\infty}(-)^k q^k B_k(y)]/(a^2+a)^{\frac{1}{2}};\ q = 2a + 1 - 2(a^2+a)^{\frac{1}{2}} \tag{4}$$

$$0 \leq y \leq 1; \quad |\arg a| \leq \pi/2, \quad a \neq 0.$$

From a numerical point of view, (4) is pathologic. If $a = 1$, the Taylor series development of $1/(y + 1)$ is slowly convergent near $y = 1$; at $y = 1$ it is divergent. However, (4) is precise and rapidly convergent in this region. The above is a striking example to show the strength of a Chebyshev expansion and the comparative weakness of a Taylor series expansion. The integral of (4) is not patho-

logic, and we find

$$\log (y+a) - \log a = 2\sum_{k=1}^{\infty} \frac{q^k}{k}\{1 - (-)^k B_k(y)\} \tag{5}$$

$$= -2\log(1-q) - 2\sum_{k=1}^{\infty} \frac{(-)^k q^k}{k} B_k(y).$$

In particular, if $y = 1$,

$$\log(a+1) - \log a = 4\sum_{k=0}^{\infty} \frac{q^{2k+1}}{2k+1}, \tag{6}$$

and if further $a = 1$, we have the interesting series

$$\log 2 = 4\sum_{k=0}^{\infty} \frac{(3-2\sqrt{2})^{2k+1}}{2k+1}. \tag{7}$$

If in the first expansion of (5) we retain N terms, then a bound for the error committed is

$$|E_N| \leq \frac{4|q^{N+1}|}{(N+1)|(1-q)|}. \tag{8}$$

It is of interest to compare (6) with the well known expression

$$\log(a+1) - \log a = 2\sum_{k=0}^{\infty} \frac{(2a+1)^{-2k-1}}{(2k+1)}. \tag{9}$$

Examination of the factor $r = q(2a+1)$ shows that the convergence of (6) is much more rapid than (9). Indeed if $a = 1$, r is quite close to its asymptotic value for large a which is $\frac{1}{3}$. Consider the error bounds for (6) and (9) which are easily derived after the manner of (8). If $a = 1$, the ratio of these bounds is about 2^{-2N-2}, and the economy of (6) is evident.

We now turn to the expansion for arc tan z. In (4), replace a by a^2, and y by y^2. Noting that $B_k(y^2) = T_{2k}(y)$, and integrating, we find that

$$\text{arc tan } y/a = 2\sum_{k=0}^{\infty} \frac{(-)^k R^{2k+1}}{2k+1} T_{2k+1}(y); \quad R = (a^2+1)^{\frac{1}{2}} - a, \tag{10}$$

$$-1 \leq y \leq 1.$$

This series converges for all values of a in the right half plane except on the line joining the points $a = \pm i$. A convenient bound for the error committed after using N terms is

$$|F_N| \leq \frac{2|R^{2N+3}|}{(2N+3)|1-R^2|}. \tag{11}$$

We also have

$$\text{arc tan } y = \pi/4 + 2\sum_{k=0}^{\infty} \frac{(-)^k(\sqrt{2}-1)^{2k+1}}{(2k+1)} T_{2k+1}\left(\frac{y-1}{y+1}\right); \quad 0 \leq y \leq \infty, \tag{12}$$

and the interesting series

$$\pi/8 = \sum_{k=0}^{\infty} \frac{(-)^k(\sqrt{2}-1)^{2k+1}}{(2k+1)}; \quad \pi/12 = \sum_{k=0}^{\infty} \frac{(-)^k(2-\sqrt{3})^{2k+1}}{(2k+1)}. \tag{13}$$

Since

$$\arctan z = \pi/2 - \arctan 1/z, \tag{14}$$

we see that (10) can be used for computation everywhere in the right half plane except when a is near the imaginary axis and $|a| \leq 1$. In this region we use the fact that

$$\arctan z = \arctan (x + iy) = \alpha + i\beta \tag{15}$$

$$\alpha = \tfrac{1}{2} \arctan \frac{2x}{1 - x^2 - y^2}; \quad \beta = \tfrac{1}{4} \log \frac{(1+y)^2 + x^2}{(1-y)^2 + x^2},$$

and the appropriate series given above. Of course, arc tan z can always be computed in this way, but (10) seems advantageous since only a single formula is needed. Our expansions are also useful in connection with the formula

$$\log z = \tfrac{1}{2} \log |z^2| + i \arctan y/x; \quad z = x + iy \tag{16}$$

For numerical purposes it may be desirable to truncate the infinite expansion and rearrange the terms so that a polynomial approximation is obtained. Details are left to the reader. However, this procedure is not necessary since the Chebyshev polynomials satisfy simple recurrence relations. Also, a previous suggestion by Clenshaw [4] is valuable. With $a = 1$, we have truncated the expansions arising from (5) and (6) for several values of N, and expressed the finite series as a polynomial in y. We find that the corresponding coefficients given in a recent volume by Hastings [5] are quite close to ours as are also the error bounds. In the latter reference, the treatment is semi-theoretical, as the development is based in part on known tabular data for the functions. The present treatment is more general and seems more pleasing as the need for tabular data is obviated.

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1. C. W. CLENSHAW, "Polynomial approximations to elementary functions," *MTAC*, v. 8, 1954, p. 143–147.
2. NBS Applied Mathematics Series, No. 9, *Tables of Chebyshev Polynomials* $S_n(x)$ and $C_n(x)$, U. S. Gov. Printing Office, Washington, D. C., 1952.
3. WOLFGANG GRÖBNER & NIKOLAUS HOFREITER, *Integraltafel. Zweiter Teil. Bestimmte Integrale*, Springer-Verlag, Vienna and Innsbruck, 1950, Formula 24, p. 112.
4. C. W. CLENSHAW, "A note on the summation of Chebyshev series," *MTAC*, v. 9, 1955, p. 118–120.
5. CECIL HASTINGS, JR., JEANNE T. HAYWARD, & JAMES P. WONG, JR., *Approximations for Digital Computers*, Princeton University Press, New Jersey, 1955. [*MTAC*, v. 9, 1955, p. 121–123.]

TECHNICAL NOTES AND SHORT PAPERS

Note upon the Numerical Evaluation of Limits of Sequences

1. Introduction. It is well known that, if a sequence s_n tends to a limit s such that

$$s = s_n + r_n$$

where

$$r_n = \frac{A}{n^2} + \frac{B}{n^3} + \cdots,$$

A, B, etc., independent of n, it is possible, using Richardson's deferred approach to the limit [1], to find a better approximation to s by calculating s_n for two different values of n and solving for s by eliminating A and neglecting terms of order less than $O(n^{-2})$; e.g.,

$$s_n' = \frac{4s_{2n} - s_n}{3} + O(n^{-3})$$

is a better approximation to s than was s_n. This formula suffers from the disadvantage that it is necessary to calculate the value of the sequence at n and also at $2n$. Furthermore this formula applies only when the remainder term is of the form indicated. It is the purpose of this note to show that, under certain conditions, an improved approximation may be derived from the values of the sequence at n and at $n + 1$.

2. Analysis. Let $r_n = Kp_n(1 + q_n)$ where $q_n = o(1)$ and p_n is a simple function of n which satisfies this relation. K is a possibly unknown constant. The meaning of the term simple in this connection may be understood from the following examples.

(1) $$r_n = \int_n^\infty \frac{dy}{y^4 + ay^2 + b}, \qquad p_n = \frac{1}{n^3},$$

(2) $$r_n = \int_{x_1}^{x_2} \frac{dx}{x^2 + n^2}, \qquad p_n = \frac{1}{n},$$

(3) $$r_n = \frac{1}{2^n}\frac{1}{n}\int_0^n \tanh\,(x + e^{-x})dx, \quad p_n = \frac{1}{2^n}.$$

The exact form of p_n is unimportant. If r_n were known exactly, there would of course be no need to carry out this process. If the order of r_n can be found, a suitable p_n can also be found. We have

$$s = s_n + Kp_n(1 + q_n)$$

$$s = s_{n+1} + Kp_{n+1}(1 + q_{n+1}).$$

Multiplying the first equation by p_{n+1} and the second by p_n, it follows that

$$s = s_n' + r_n'$$

where

$$s_n' = \frac{p_n s_{n+1} - p_{n+1} s_n}{p_n - p_{n+1}}$$

$$r_n' = \frac{K p_n p_{n+1}(q_{n+1} - q_n)}{p_n - p_{n+1}}.$$

Thus a better approximation to s will have been obtained if

$$\left|\frac{r_n'}{r_n}\right| = \left|\frac{q_{n+1} - q_n}{p_{n+1} - p_n} \cdot \frac{p_{n+1}}{1 + q_n}\right| < 1.$$

This inequality must be verified before we are able to use the method. In section 3, a full discussion will be given of the calculation of $\pi/2$.

3. Calculation of $\pi/2$. A formula for $\pi/2$ is $\pi/2 = \lim s_n$, where

$$s_n = \int_0^n \frac{dx}{1 + x^2}.$$

It follows that

$$r_n = \int_n^\infty \frac{dx}{1 + x^2} = \int_0^{1/n} \frac{dy}{1 + y^2}.$$

Clearly a suitable p_n is $1/n$ and

$$r_n = \int_0^{1/n} \frac{dy}{1 + y^2} = \frac{1}{n}(1 + q_n) < \frac{1}{n}$$

as

$$q_n = -n \int_0^{1/n} \frac{y^2 dy}{1 + y^2} > -\frac{1}{3n^2}$$

and

$$1 + q_n > 1 - \frac{1}{3n^2}.$$

Also

$$s_n' = (n + 1)s_{n+1} - n s_n = s_{n+1} + n(s_{n+1} - s_n).$$

(It will be observed that $p_n = 1/n$ gives rise to a particularly convenient form for s_n'.) Furthermore

$$\frac{p_n - p_{n+1}}{p_{n+1}} = \frac{1}{n}.$$

It may be shown that

$$q_{n+1} - q_n = \int_0^1 \frac{2n + 1}{n^2(n + 1)^2} \frac{z^2 dz}{\{1 + (z/n)^2\}\{1 + (z/n + 1)^2\}} < \frac{2n + 1}{3n^2(n + 1)^2}.$$

It follows that

$$\left|\frac{r_n'}{r_n}\right| < \frac{2n^2 + n}{(3n^2 - 1)(n + 1)^2}$$

$$r_n' < \frac{2n + 1}{(3n^2 - 1)(n + 1)^2}$$

(r_n' and r_n are both positive). It follows therefore that the convergence of the method is fairly rapid. If $n = 1$ the remainder ratio is 3/8, whereas if $n = 10$ it is .006.

$$s_{10} = 1.47113$$

$$s_{11} = 1.48014$$

$$s_{10}' = s_{11} + 10(s_{11} - s_{10})$$

$$= 1.5702.$$

It will be observed that, because $s_{11} - s_{10}$ is multiplied by 10, a decimal place of accuracy is lost. Also the figures given for s_{10} and s_{11} may be in error by .5 units of the last place, and so $s_{10}' = 1.5702 \pm .0001$, it being possible for $s_{11} - s_{10}$ to have an error 1 in the last place. Also $r_{10}' < 21/299 \times 121 = .0006$ and is positive. Thus $1.5702 - .0001 < \pi/2 < 1.5702 + .0001 + .0006$ or $1.5701 < \pi/2 < 1.5709$, comparing well with the true value 1.5708. The agreement is remarkably good for the comparative roughness of the approximation.

The first draft of this paper was written while the author was on the staff of the Royal Military College of Science, Shrivenham, and was communicated by kind permission of the Commandant.

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1. SIR HAROLD JEFFREYS & BERTHA SWIRLES JEFFREYS, *Methods of Mathematical Physics*, Cambridge Univ. Press, New York, 1956, p. 265.

Factors of Fermat Numbers

The writer has prepared a multiple precision routine for the SWAC which tests numbers of the form $N = k \cdot 2^n + 1$ for primeness. If N is prime, it is then tested to find whether it divides any Fermat number $F_m = 2^{2^m} + 1$. The running time for either test is about a minute and a half for n near 500, and about seven minutes for n near 1000. There is also a preliminary sieve routine which examines N for small factors. If there is no small factor, the smallest positive number a for which $(a/N) = -1$ is found. The congruence $a^{(N-1)/2} \equiv -1 \pmod{N}$ is then a necessary and sufficient condition for primeness, at least if $k < 2^n$ [1].

During the period September–November 1956, the cases $k = 3, 5, 7$ were run for $n < 1024$, and the odd values of k from 9 to 57 were run for $n < 512$. Some isolated larger values of k have also been used. The cases for k up to 17 and the other results listed below have been checked by a second run. The work is con-

tinuing. The operation of the SWAC has been handled, for the most part, by John L. Selfridge.

Fourteen new factors of Fermat numbers have been found, namely

$$21 \cdot 2^{41} + 1 \mid F_{39}, \quad 29 \cdot 2^{57} + 1 \mid F_{55}, \quad 9 \cdot 2^{67} + 1 \mid F_{63},$$
$$7 \cdot 2^{120} + 1 \mid F_{117}, \quad 5 \cdot 2^{127} + 1 \mid F_{125}, \quad 17 \cdot 2^{147} + 1 \mid F_{144},$$
$$1575 \cdot 2^{167} + 1 \mid F_{160}, \quad 3 \cdot 2^{209} + 1 \mid F_{207}, \quad 15 \cdot 2^{229} + 1 \mid F_{226},$$
$$29 \cdot 2^{231} + 1 \mid F_{228}, \quad 21 \cdot 2^{276} + 1 \mid F_{265}, \quad 7 \cdot 2^{290} + 1 \mid F_{284},$$
$$7 \cdot 2^{320} + 1 \mid F_{316}, \quad 27 \cdot 2^{455} + 1 \mid F_{452}.$$

Also, the number $N = k \cdot 2^n + 1$ was found to be prime in just the following cases with $k = 3, 5, 7$ and $20 < n < 1024$.

$k = 3$: $n = 30, 36, 41, 66, 189, 201, 209, 276, 353, 408, 438, 534.$

$k = 5$: $n = 25, 39, 55, 75, 85, 127.$

$k = 7$: $n = 26, 50, 52, 92, 120, 174, 180, 190, 290, 320, 390, 432, 616, 830.$

After the project is completed, it is planned to publish a list of all the primes found.

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The work reported was sponsored by the Office of Naval Research.

1. L. E. DICKSON, *History of the Theory of Numbers*, Carnegie Inst. of Washington, v. 1, 1919, p. 92.

REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

1[A].—H. S. UHLER, "Exact values of 996!, and 1000!, with skeleton tables of antecedent constants," *Scripta Mathematica*, v. 21, 1955, p. 261–268.

This paper gives the values of 996! and 1000! without the terminal zeros of which there are 246 and 249 respectively. There is also given the exact value of $(996! + 1)/997$, together with a frequency census of the digits 0–9 in each of these three large numbers. The last mentioned number is a "Wilson quotient" and the fact that it is an integer constitutes a check on the work. Certain partial products of 25 consecutive integers beyond 750 are also given.

The work was done by desk calculator as far as 750! and then finished on the UNIVAC. The reviewer has compared the value of 1000! with the result obtained directly on the SWAC by Kenneth Ralston. The agreement was perfect.

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2[A].—S. C. NICHOLSON & J. JEENEL, "Some comments on a NORC computation of π," *MTAC*, v. 9, 1955, p. 162–164.

This gives π, 3089D. It also includes frequency counts of the digits in π.

C. B. T.

3[B, C, D].—E. H. NEVILLE, editor, *Rectangular-Polar Conversion Tables*. Designed and compiled by E. H. Neville, Roy. Soc. Math. Tables, v. 2, Camb. Univ. Press, New York, 1956, xxxii + 109 p., 28 cm. Price $5.50.

The stated purpose of this set of double-entry conversion tables is to provide values of r and θ corresponding to pivotal values of x and y. To this end, for each positive integer M not exceeding 105 there appears a table of r and θ in degrees to 13D for $x = M$ and $y = 1(1)M$, followed by a companion table of $\ln r$ and θ in radians to 15D for $y = M$ and $x = M(1)105$.

E. H. Neville has written an extensive introduction, which includes an account of the evolution of these tables, together with a full description of the elaborate checking procedures employed, which insure the attainment of the exemplary tabular accuracy exhibited earlier in the publications of the British Association Mathematical Tables Committee, on which Professor Neville served as chairman for sixteen years.

As explained in the Introduction, the calculation of the polar angle θ was made to depend basically on the Haros property of Farey series and an auxiliary table of the denominators of the Farey series of order 105 is included to expedite interpolation. A scholarly historical account of Farey series appears in R. C. Archibald's review [*MTAC*, RMT **881**, v. 5, 1951, p. 135–139] of the first volume of the present series of tables.

Both direct and inverse interpolation in all tabulated quantities, to the full tabular accuracy, are illustrated by detailed numerical examples. (It should be remarked that a desk calculator is almost a necessity in such interpolation.)

Additional supplementary tables give $\ln x$, $x = 1(1)160$, 15D, and 15D values of the first 20 multiples of $\ln 10$. The volume is concluded with page indexes for r in the range 70.0 to 109.9 and $\ln r$ in the interval 4.400 to 4.709 designed to aid in the location of close tabular approximations, prior to interpolation by series. A small table of "adjusting factors" and their natural logarithms to 15D is included to bring general values of r and $\ln r$ within the prescribed range.

The versatility of the conversion tables is illustrated by their application to the calculation of the principal value of $(e + i\gamma)^{\tau}$ to about eleven significant figures in both rectangular and polar form.

The application of the tables to the conversion of oblique Cartesian coordinates to polar coordinates is also illustrated.

An extensive bibliography, compiled by J. C. P. Miller, has been included. This lists—with appropriate comments—the most important earlier tables in this field and also enumerates the works used in the preparation of the present compilation.

The high standard of typographical excellence characteristic of the earlier publications of both this committee and their predecessors has been maintained. The present set of tables, in providing reliable key values for future tabulations, as well as very accurate working data, constitutes a valuable addition to the growing literature of such numerical information.

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4[C].—HANS J. MAEHLY, *Rational Approximation*, Cecil Hastings' "Approximation Newsletter," October 25, 1956, Note-16.

Rational approximations for

$$\log_2(x),\ \tfrac{1}{2} \le x \le 1,\ 10\text{D};\quad 2^x,\ -\tfrac{1}{2} \le x \le \tfrac{1}{2},\ 10\text{D},$$

and

$$\arctan x,\ -1 \le x \le 1,\ 9\text{D}.$$

C. B. T.

5[D].—R. B. HORGAN, "Radix tables for $\sin x$ and $\cos x$," *MTAC*, v. 10, 1956, p. 164–166.

$\sin x$ and $\cos x$ for $x = a \cdot 10^k$ degrees, $a = 1(1)9$, $k = -3(1)1$, 20D. Also included are summation formulas for sin and cos of sums of 2, 3, 4, and 5 angles.

C. B. T.

6[I].—G. A. CHISNALL, "A modified Chebyshev-Everett interpolation formula," *MTAC*, v. 10, 1956, p. 66–73.

Coefficients for obtaining appropriately modified differences $N^j (j = 2(2)10)$ in terms of ordinary central differences $\delta^j (j = 2(2)10)$ are given to 12D; so, too, are the corresponding multipliers β_i^j in the interpolation formula of the title, which is of the form

$$\begin{aligned} f(m) &= f(0) + m[f(1) - f(0)] + \sum \{\beta_0^j C_{2j+1}\{(2 - 2_m)\theta_j\} + \beta_1^j C_{2j+1}(2m\theta_j)\} \\ &= f(0) + m[f(1) - f(0)] \\ &\quad + \sum k_j \{N_0^{(2j)} C_{2j+1}\{(2 - 2m)\theta_j\} + N_1^{(2j)} C_{2j+1}(2m\theta_j)\}, \end{aligned}$$

where

$$k_j = 2^{-2j-1} \sec^{2j+1} \theta_j / (2_{j+1})!,\ \theta_j = \cos(\pi/(4j + 2)) \text{ and } C_j(2x) = 2\cos(j \arccos x).$$

J. T.

7[I, X].—P. C. HAMMER, O. J. MARLOWE, & A. H. STROUD, "Numerical integration over simplexes and cones," *MTAC*, v. 10, 1956, p. 130–137.

Let $q_m(x) = q_m^n(x)$, $m = 1, 2, \cdots$, denote the polynomials orthogonal with respect to x^n in the interval $(0, 1)$. These are given for $m = 1(1)5$, $n = 1, 2$ and for $m = 1(1)4$, $n = 3$. The zeros of the q_m^n are given to 18S and so are the corresponding weights b_j in the approximate quadrature formula

$$\int_0^1 x^n g(x)dx \doteq \sum b_j g(x_j).$$

Combination of two of these formulas gives a quadrature formula of the form

$$\begin{aligned} \int_\Delta f d\sigma &= \sum\sum b_j a_k f(x_j, x_j y_k) \\ &= \sum\sum w_{jk} f(x_j, x_j y_k) \end{aligned}$$

which is exact for a polynomial of degree 7, when Δ is a triangle (normalized to $(0, 0)$, $(1, 1)$, $(1, -1)$). The sixteen weights and abscissae are given to 18S.

J. T.

8[I, X].—K. S. KUNZ, "High accuracy quadrature formulas from divided differences with repeated arguments," *MTAC*, v. 10, 1956, p. 87–90.

Let $S_r = \sum_{i=1}^{r} i^{-1}$ for $r > 0$. There is a table of the rational numbers, $2[S_p - S_{n-p}]\binom{n}{p}^2$ and the integers $\binom{n}{p}^2$ for $p = 0(1)6$, $n = 1(1)p$. The values of the integers $(2n + 1)!/(n!)$ are given for $n = 1(1)6$. There are various minor typographical faults in the paper.

J. T.

9[J, X].—HERBERT E. SALZER, "Formulas for the partial summation of series," *MTAC*, v. 10, 1956, p. 149–156.

The table lists coefficients $A_m(n)$ in the "partial" summation formula $S_n \doteq \sum_{m=4}^{10} A_m(n)S_m$, $m = 4(1)10$, $n = 11(1)50(5)100(10)200(50)500(100)1000$. Exact as rational fractions. See also Review 10 below.

C. B. T.

10[J].—R. B. HORGAN, "Coefficients for the partial summation of series," *MTAC*, v. 10, 1956, p. 156–162.

This table lists $A_m(n)$ (see Review 9, this issue) for $m = 4(1)10$ and $n = 11(1)50(5)100(10)200(50)500(100)1000$, 15D.

C. B. T.

11[K].—HARVARD UNIVERSITY, Computation Laboratory, *Annals*, v. 35, *Tables of the Cumulative Binomial Probability Distribution*, Cambridge, Mass., Harvard University Press, 1955, lxi + 503 p., 27.3 cm. Price $8.00.

These tables give cumulative binomial probabilities

$$E(n, r, p) = \sum_{x=r}^{n} \binom{n}{x} p^x(1 - p)^{n-x}, \quad 0 \leqq r \leqq n, \quad \text{to 5D}$$

for $p = .01(.01).5$, 1/16, 1/12, 1/8, 1/6, 3/16, 5/16, 1/3, 3/8, 5/12, 7/16, and $n = 1(1)50(2)100(10)200(20)500(50)1000$. Of course cumulative binomial probabilities for $p = .51(.01)1$ are immediately available from the tables since $E(n, r, p) = 1 - E(n, n - r + 1, 1 - p)$.

The present tables should be compared with the Army Ordnance "Tables of the Cumulative Binomial Probabilities" [1] which give cumulative binomial probabilities for $p = .01(.01).5$ and $n = 1(1)150$ to 7D. The Harvard tables therefore extend previously available tables of binomial probabilities to some useful higher values of n, while leaving some important gaps in sample sizes $n \leq 150$ and recording fewer decimal places as compared to the Ordnance tables. Furthermore, interpolation in the tables for missing values of n is not necessarily

a pleasant pastime. The additional probabilities for specific fractional values of p such as $p = 1/12, 1/6$, etc., are a good feature of the new Harvard tables and will serve a use, perhaps to dice throwers, certainly for game probabilities and indeed for computations on many more or less natural phenomena.

In using the Harvard tables, one finds that they are not consecutive for the sample size n; rather they are divided into six parts on the chance of success in a single trial, p, as follows: (1) Table I, $.01 \leqq p \leqq 1/12$, (2) Table II, $.09 \leqq p \leqq 1/6$, (3) Table III, $.17 \leqq p \leqq 1/4$, (4) Table IV, $.26 \leqq p \leqq 1/3$, (5) Table V, $.34 \leqq p \leqq 5/12$, and (6) Table IV, $.42 \leqq p \leqq 1/2$. For fixed n, the tables are further subdivided into subtables. A user of the tables will nevertheless become accustomed to the required manipulations rapidly.

Much is to be said for the Introduction which consists of: (1) the mathematical characteristics of the cumulative binomial probability distribution, (2) methods of preparation of the tables, (3) methods of interpolation for the tables, and (4) the very excellent section IV on applications by Frederick Mosteller.

It is considered somewhat unfortunate by the reviewer that here is a large new table which does not encompass readily or include in one volume or set all binomial probabilities tabulated to date. One will still want to use these tables along with other existing tabulations.

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1. L. E. SIMON & F. E. GRUBBS, *Tables of the Cumulative Binomial Probabilities*, U. S. Gov. Printing Office, Washington, D. C., 1954.

12[K].—E. C. FIELLER, T. LEWIS, & E. S. PEARSON, *Correlated Random Normal Deviates*, Tracts for Computers, No. XXVI, Cambridge Univ. Press, New York, 1955, xv + 60 p., 23.2 cm. Price $2.00.

"These tables give in effect 3000 random pairs from each of nine bivariate normal distributions with respective correlation coefficients .1, .2, .3, ···, .9. They have been compiled from Wold's table of random normal deviates [1].

"The data are set out in the form: ith line: $x_{0i}, x_{1i}, x_{2i}, \cdots, x_{9i}$ ($i = 1, 2, \cdots,$ 3000), where the ten values are random members of normal populations with means 0 and variances 1, and with correlation coefficients $\rho_t = t/10$ between x_0 and x_t ($t = 1, 2, \cdots, 9$)." All values are given to 2D.

In all, 25,000 lines were computed in which the x_{0i} were the 25,000 random deviates of Wold's table and values of x_{ti} were obtained as suggested by Wold (p. xii) by evaluating, $x_{ti} = \rho_t x_{0i} + (1 - \rho_t^2)^{\frac{1}{2}} z_{ti}$ in which the z_{ti} are a new random arrangement of the x_{0i}. The 3000 lines printed were selected according to the x_{0i} by making a random choice of 6 of the 20 pages of 500 which constitute the 3rd and 4th of the 5 blocks of Wold's tables. These blocks passed all of Wold's tests for randomness.

At the foot of each column of 50 entries there are given for the x_{ti} in that column, $\sum x_{ti}$ to 2D, $\sum x^2_{ti}$ to 4D, $\sum x_{0i}x_{ti}$ to 4D, and the sample correlation coefficients r_{0t} to 4D. In addition in table 1 for the 3000 values of each x_t, the ranges (2D), means (4D), standard deviations (4D), and third and fourth standard cumulants (4D) are listed.

The authors describe the application of four tests of randomness which were used on the complete set, on six sets of 500, and on sixty sets of 50 into which the table was divided. Performance on all of these tests seemed quite satisfactory.

C. C. C.

1. HERMAN WOLD, *Random Normal Deviates*, Tracts for Computers, No. XXV, Cambridge Univ. Press, New York, 1948.

13[K].—E. J. HANNAN, "An exact test for correlation between time series," *Biometrika*, v. 42, 1955, p. 316–326.

This article is concerned with tests for correlation between two time series x_t and y_t with serially correlated normal residuals. The estimates compared are: (1) the partial correlation between x_{2t} and y_{2t} when the effects of $(y_{2t-1} + y_{2t+1})$, x_{2t-1} and x_{2t+1} have been removed; (2) the ordinary correlation coefficient r between the two series, and (3) the partial correlation between x_t and y_t given x_{t-1} and y_{t-1}. The asymptotic efficiencies of these statistics are compared under the conditions: (a) the residual process from the regression of y_t and x_t is independent of the x_t process and comes from a Gaussian Markov process; (b) the two series are Markovian and are correlated through correlated errors; (c) same as (b) but with second order autoregression. The paper shows statistic (3) to lead to the asymptotically most efficient test for conditions (a), (b), and (c), except for some cases under (c) where the first partial correlation of the x_t process is high and positive. The criterion used for comparison requires the evaluation of the quantities:

$$\frac{(1 + p_1p_2)(1 - p_2^2)}{(1 - p_1p_2)(1 - p_1^2)}$$

and

$$\frac{1}{2}\,\frac{(1 - p_1^2p_2^2)(1 + p_1p_2)^2}{(1 - p_1^4)(1 - p_2^4)}$$

which are tabulated to 2D for $p_1, p_2 = -.8(.2).8$ and

$$\frac{(1 + p_1p_2 - b)^2[1 + p_1p_2 - b(p_1^2 + p_1p_2)][1 - p_1p_2 - b(p_1p_2 - p_1^2)]}{2(1 - p_1^4)(1 - p_2^2)(1 - b)[1 + p_2^2 - b(1 - p_2^2)]}$$

which is tabulated to 2D for $p_1 = p_2 = .4$ and all combinations of $p_1, p_2 = .6, .8$ for $b = -.6(.2).6$.

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14[K].—G. E. NOETHER, "Use of the range instead of the standard deviation," Amer. Stat. Assn., *Jn.*, v. 50, 1955, p. 1040–1055.

This expository article summarizes methods which use the range instead of the standard deviation: (1) to obtain confidence intervals for the mean μ of a normal population, the difference of two means, and (2) to estimate the standard deviation σ of a normal population. Table 1 gives for $N = 2(1)100$ the appropriate subsample size, the necessary factors to 3S to obtain an unbiased estimate of σ and to 2D to find 90% and 98% upper and lower confidence limits. This table is

based on the optimum procedure given by Grubbs and Weaver [1]. It is pointed out that the loss in efficiency due to the use of equal subsamples is slight compared to the gain in computational ease. For use in obtaining confidence intervals for the mean and difference of means, Tables I and II of Jackson and Ross [2], which were derived from earlier tables by Lord, are recommended.

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1. F. E. GRUBBS & C. L. WEAVER, "The best unbiased estimate of population standard deviation based on group ranges," Amer. Stat. Assn. *Jn.*, v. 42, 1947, p. 224–241.
2. J. E. JACKSON & E. L. ROSS, "Extended tables for use with the 'G' test for means," Amer. Stat. Assn., *Jn.*, v. 50, 1955, p. 416–433.

15[K].—D. P. BANERJEE, "A note on the distribution of the ratio of sample standard deviations in random samples of any size from a bivariate correlated normal population," Indian Soc. Agricultural Stat. *Jn.*, v. 6, 1954, p. 93–100.

For samples of N from a normal bivariate universe, the author has tabulated to 3S the upper 80%, 90%, 95%, and 99% points of the distribution of the ratio of the two sample standard deviations for $N = 3(1)30$ and $\rho = 0(.1).9$, where ρ is the universe coefficient of correlation. For $\rho = 0$ the values given in a high proportion of cases are one less in the third significant figure than the 3S square root of the corresponding variance ratio, F, given in the standard tables.

C. C. C.

16[K].—D. A. S. FRASER & IRWIN GUTTMAN, "Tolerance regions," *Annals Math. Stat.*, v. 27, 1956, p. 162–179.

Tables in this article give factors for obtaining certain tolerance regions for univariate and multivariate normal distributions.

The tolerance intervals (regions) are termed similar β-expectation if the average probability content of the interval (region) is β. The appropriate factors are given to 4S for five cases: univariate normal with unknown mean and variance, univariate normal with known variance, bivariate, trivariate, and quadri-variate normal with unknown means, and variance-covariance matrices for $\beta = .75, .9, .95, .975, .99, .995$ for sample size $n = 2(1)31, 41, 61, 121, \infty$ for the univariate cases and n increased by one less than the number of variates for the multivariate cases.

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17[K].—J. ARTHUR GREENWOOD & DAVID DURAND, "The distribution of length and components of the sum of n random unit vectors," *Annals Math. Stat.*, v. 26, 1955, p. 233–246.

The n random unit vectors of the title of the paper are in Euclidean 2-space and are drawn independently from a distribution with probability density for the

angle ξ, measured from some fixed direction, $g(\xi) = [\exp\{k\cos(\xi - \alpha)\}]/2\pi I_0(k)$. Mises [1] introduced this distribution of points on a circle as the distribution (under certain requirements with respect to continuity of derivatives) for which the maximum likelihood estimate of the center of gravity of the distribution is the center of gravity of the sample points. Gumbel, Greenwood, and Durand [2] in 1953 gave tables of the distribution function of $g(\xi)$ for various values of k and a table to facilitate calculation of the maximum likelihood estimate of k. The present paper gives functional forms for the distribution of $V = \sum_{r=1}^{n} \cos \xi_r$ for $k = 0$ and for $\alpha = 0$, the joint distribution of V and $W = \sum_{r=1}^{n} \sin \xi_r$ for $\alpha = 0$, and the distribution of $R = \sqrt{V^2 + W^2}$.

The tables in the present paper deal with the distribution of R when $k = 0$, i.e., with samples from the uniform angular distribution, $g(\xi) = 1/2\pi$. Table 1 gives the probability that $R \leq r$, $P(r, n) = r\int_0^\infty [J_0(x)]^n J_1(rx)dx$ to 5D for $n = 6(1)24$, $r = .5(.5)12.(1)n$. $P(n, n) = 1$ and the value 1.00000 is entered only for the lowest value of r for which it is appropriate. Table 2 gives 95th and 99th percentiles of the distribution of R and of various functions of r. The solutions of $P(r, n) = .95$ and $P(r, n) = .99$ are given to 3D. Corresponding values of r/n are given to 4D, r^2 to 2D, $z = r^2/n$ to 4D. The two limiting values of z for $n \to \infty$ are given. The authors describe the computational procedures used and give reasons for believing the tables accurate to the number of places given, except that $P(r, n)$ for $n = 6$ may be in error by 1 in the fifth place and z in error by 2 in the fourth place. Table 3 compares $P(r, n)$ with the approximation $1 - \exp(-r^2/n)$ and with two other approximations. Table 4 compares percentage points with various approximations to them. An example of a test of significance of R is given with n outside the range of the tables and, therefore, an approximation to a percentage point of $P(r, n)$ is calculated.

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1. R. v. Mises, "Über die 'Ganzzahligkeit' der Atomgewichte und verwandte Fragen," *Physikalische Zeitschrift*, v. 19, 1918, p. 490–500.
2. E. J. Gumbel, J. A. Greenwood, & D. Durand, "The circular normal distribution: theory and tables," Am. Stat. Assn. *Jn.*, v. 48, 1953, p. 131–152. [*MTAC*, v. 8, RMT 1151, 1954, p. 19.]

18[K].—Peter Ihm, "Ein Kriterium für zwei Typen zweidimensionaler Normalverteilungen," *Mitteilungsblatt für Math. Stat.*, v. 7, 1955, p. 46–52.

A significance test is constructed for the composite hypothesis H_0: $\sigma_1 = \sigma_2$; $\rho = 0$ for a bivariate normal distribution p and a sample size N. The alternative H_1 is $\sigma_1 \neq \sigma_2$ or $\rho \neq 0$ or both. Let $L = p_0/p_1$ be the likelihood ratio. The test function used is $Z = L^{2/N}$. In order to test H_0 against H_1, the author calculates the probability $P(Z \leq Z_0|H_0) = \alpha$. If $Z \leq Z_0$, then H_0 is rejected and H_1 is accepted. If $Z > Z_0$, then H_1^* is rejected and H_0^* accepted, where H_1^* and H_0^* stand respectively for $\rho^2 \geqq \rho^{*2}$. A table gives Z_0 to 3D as a function of $N = 3(1)30$, 40, 60, 120 for $\alpha = .01$ and $\alpha = .05$.

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19[K].—F. M. LORD, "Nomograph for computing multiple correlation coefficients," Amer. Stat. Assn., *Jn.*, v. 50, 1955, p. 1073–1078.

The author presents a nomograph for obtaining the multiple correlation coefficient $R_{1.23} = \{(r^2_{12} + r^2_{13} - 2r_{12}r_{13}r_{23})/(1 - r^2_{23})\}^{\frac{1}{2}}$, where r_{12}, r_{13}, and r_{23} are the ordinary coefficients of correlation.

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20[K].—MOTOSABURA MASUYAMA, "Tables of two-sided 5% and 1% control limits for individual observations of the r-th order," *Sankhyā*, v. 15, 1955, p. 291–294.

The 95% and 99% confidence limits here found for the rth order statistic in a sample of n from any continuous distribution are obtained by finding 2.5% and 97.5% (or 0.5% and 99.5%) points of the corresponding incomplete β distribution, using a well known relation. These limits are tabulated to 5S for $n = 3(1)6, 8, 10$ and $r = 1(1)n$. Most of the values listed could be copied directly from the Thompson tables [1]; it is not stated how the ones not found there were obtained. In addition in the case where the distribution sampled is $N(0, 1)$, the probability integral variates tabled are converted to standardized normal deviates to 4S apparently by linear interpolation in the Kelley tables [2], which, with the Thompson tables, are listed in the author's bibliography.

C. C. C.

1. C. M. THOMPSON, "Tables of percentage points of the incomplete beta-function and of the χ^2 distribution," *Biometrika*, v. 32, 1941, p. 151–181. Also Table 16 in *Biometrika Tables for Statisticians*, vol. 1, Cambridge, 1954. [*MTAC*, v. 1, 1943, RMT 99, p. 76–77.]
2. *The Kelley Statistical Tables*, Cambridge, Mass., 1948. [*MTAC*, v. 1, 1944, RMT 130, p. 151–152.]

21[K].—M. R. SAMPFORD, "The truncated negative binomial distribution," *Biometrika*, v. 42, 1955, p. 58–69.

This paper is concerned with estimating the parameters p and k of the negative binomial distribution, $P(r) = \binom{k+r-1}{r} p^r(1 + p)^{-(k+r)}$, $(r = 0, 1, \cdots; p, k > 0)$, when truncation has resulted in elimination of the class corresponding to $r = 0$. Moment estimating equations obtained by obtaining the population mean and variance to corresponding sample values and maximum likelihood estimating equations are also derived.

Explicit solutions in terms of elementary functions are not possible for either the moment or the maximum likelihood estimating equation, and iterative procedures must be employed. The author gives illustrative examples to demonstrate that the moment estimating equations are simpler to solve than the maximum likelihood equations. A table of efficiencies of the moment method is given to 3S for $k = 0.5, 1(1)5$ with means of 0.5, 1.0, 2.0, 5.0. To facilitate the iterative solution of both moment and maximum likelihood equations, the function $\varphi(x) = -x \log_e x/(1 - x)$, which occurs in both, is tabulated to 4D for $x = 0(.01)0.99$.

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22[K].—J. L. HODGES, JR., "On the non-central beta-distribution," *Annals Math. Stat.*, v. 26, 1955, p. 648–653.

Nicholson [1] has derived a closed expression for B, the noncentral beta-distribution in case b is an integer:

$$B(x; a, b, \lambda) = 1 - e^{-\lambda x}\{I_{1-x}(a, b) + (1 - x)^a \sum_{j=1}^{b-1} [x(1 - x)\lambda]^j (P_{j/j})\}$$

where

$$P_j = \sum_{k=0}^{b-j-1} \left[(-1)^k \binom{b - j - 1}{k} \frac{(a + b - 1)(a + b - 2)\cdots(a + j)}{(b - j - 1)!(a + j + k)}\right](1 - x)^k,$$

and $I_x(a, b)$ is the beta-distribution. The author proves that $P_j = \sum_{t=0}^{b-j-1} \binom{A+t}{t} x^t$, $A = a + j - 1$. A table of $\binom{A+t}{t}$ to 7S is provided for $A = .5(1)19.5$, $t = 1(1)18$. The author compares these direct methods for computing B with Tang's recursion formula [2].

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1. W. L. NICHOLSON, "A computing formula for the power of the analysis of variance," *Annals Math. Stat.*, v. 25, 1954, p. 607–610.
2. P. C. TANG, "The power function of the analysis of variance test with tables and illustrations of their use," *Stat. Res. Memoirs*, v. 2, 1938, p. 126–149.

23[K].—L. H. MILLER, "Table of percentage points of Kolmogorov statistics," Amer. Stat. Assn., *Jn.*, v. 51, 1956, p. 111–121.

Let $S_n(X)$ be the observed cumulative distribution of a random sample of n observations from a population having a continuous cumulative distribution $F(X)$. Let $D_n = \max\{S_n(X) - F(X)\}$ and $D_n^* = \max|S_n(X) - F(X)|$. Table 1 gives for $\alpha = .005, .01, .025, .05, .10$ and $n = 1(1)100$ values of ϵ to 5D such that $\alpha = \text{Prob.}\ (D_n \geq \epsilon)$. For $\alpha < .1$, $P = \text{Prob.}\ \{D_n^* \leq \epsilon\}$ is close to $1 - 2\alpha$ and the table also gives in its heading $P = .99, .98, .95, .9$.

The author believes that the greatest error in Table 1 (considering ϵ as a function of n and α) does not exceed one unit in the fifth decimal place. For $n \leq 20$ an exact formula was used, and Table 3 gives a number of checks of the asymptotic formula for $n \geq 20$. Table 2 gives an illustration of part of the computing technique.

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24[K].—C. W. TOPP & F. C. LEONE, "A family of J-shaped frequency functions," Amer. Stat. Assn., *Jn.*, v. 50, 1955, p. 209–219.

The authors consider a family of cumulative frequency functions defined by

$$(1) \qquad \begin{cases} F(x) = \dfrac{a}{b^{2r}}(2bx - x^2)^r + (1 - a)\dfrac{x}{b}, & 0 \leq x \leq b < \infty, \\ F(x) = 0, & x < 0, \\ F(x) = 1, & x > b, \end{cases}$$

where $0 < r < 1$, and $0 < a \leq 1$. The paper takes its title from the fact that the corresponding family of frequency functions obtained by differentiation of (1) are J-shaped.

The parameters a and r are expressed as functions of α_3 and δ where α_k is the k-th standard moment and $\delta = (2\alpha_4 - 3\alpha_3^2 - 6)(\alpha_4 + 3)^{-1}$. To facilitate graduation of observed data by means of (1) in the special case where $b = 1$, α_3^2 and δ are tabulated to 3D for the arguments a and r, with $r = .01, .02, .05, .08, .09, .10(.05).95$ and $a = .05(.05)1.00$. The tabled information is also presented graphically in the form of an (α_3^2, δ) chart.

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25[K].—SIGEITI MORIGUTI, "Confidence limits for a variance component," Union of Japanese Scientists and Engineers, *Reports of Statistical Application Research*, v. 3, 1954, p. 29–41.

If, as in the case of an analysis of variance with random effects, one has a mean square V_1 with f_1 degrees of freedom, whose expected value is $\sigma^2 + nv$, such that $f_1V_1/\sigma^2 + nv$ obeys a χ^2 distribution, and also has an independent mean square V with f degrees of freedom, whose expected value is σ^2 such that fV/σ^2 obeys a χ^2 distribution, it is of interest to determine confidence intervals for the variance component v, n being a known constant and σ^2 a nuisance parameter. The author derives his approximation formulas for $100(1 - \alpha)\%$ confidence limits in which for each limit two parameters enter linearly. These parameters are tabulated for $\alpha = .1$ to 4S or 3D for $f = 6(2)12, 15, 20, 30, 60$ and $f_1 = 1(1)6(2)12, 15, 20, 30, 60, \infty$. Comparisons are made with previously obtained approximations which favor the present one.

C. C. C.

26[K].—SHOJI URA, "A table of the power function of the analysis of variance tests," Union of Japanese Scientists and Engineers, *Reports of Statistical Application Research*, v. 3, 1954, p. 23–28.

The author extends the inverse tables of E. Lehmer [1] for probabilities of errors of the second kind in the variance ratio test ordinarily used in the analysis of variance. He develops a formula for the necessary power function which he credits to J. Yamauti (apparently hitherto unpublished) and employs it to tabulate values of the quantity $\psi = [(f_1 + 1)/f_1]^{\frac{1}{2}}\varphi$, where φ is the quantity tabled by Lehmer and introduced by P. C. Tang [2], for which the significance level is .05 and the probability of an error of the second kind is .1. Values are given to 2D for the degrees of freedom, $f_1 = 1(1)10, 12, 15, 20, 24, 30, 60, 120, \infty$ and $f_2 = 2(2)20, 24, 30, 40, 60, 120, \infty$.

C. C. C.

1. EMMA LEHMER, "Inverse tables of probabilities of errors of the second kind," *Annals Math. Stat.*, v. 15, 1944, p. 388–398.
2. P. C. TANG, "The power function of the analysis of variance tests with tables and illustrations of their use," *Stat. Res. Memoirs*, v. 2, 1938, p. 126–194 + tables.

27[K].—MICHIO TAKASHIMA, "Tables for testing randomness by means of lengths of runs," *Bull. Math. Stat.*, v. 6, 1955, p. 17–23.

In testing for randomness an ordered arrangement of $(m + n)$ objects of two kinds (say m A's and n B's), one may use as test criterion the length of the longest run, or of the longest A-run. The author has tabulated the critical run-lengths for tests based on these criteria.

Let $Q(t)$ be the probability that there appears at least one run (of A's or B's) of length t or longer. Let $Q_1(t)$ be the probability that there appears at least one A-run of length t or longer. Let $t_\alpha(t_\alpha)$ be the smallest integer such that $Q(t) \leq \alpha$; $(Q_1(t) \leq \alpha)$.

The tables give t_α and (t_α) for $\alpha = .01, .05$ and for $m, n = 1(1)25$. Calculations were based on an investigation by Mood [1].

Misprints in the introduction:

$$\text{line 9, } \textit{read } m = \sum_{i=1}^{m} ir_i,\ n = \sum_{j=1}^{n} js_j\text{; line 10, } \textit{read } u = r + s.$$

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1. A. M. MOOD, "The distribution theory of runs," *Annals Math. Stat.*, v. 11, 1940, p. 367–392.

28[K].—J. F. SCOTT & V. J. SMALL, "A numerical investigation of least squares regression involving trend-reduced Markoff series," Roy. Stat. Soc., *Jn.*, s. B., v. 17, 1955, p. 105–114.

Tables 1 and 2 give for a number of values of two parameters, ρ_ϵ and ρ_x (representing serial correlations of independent and residual variables), factors which facilitate the computation in the formula for the asymptotic variance of an estimate of a regression slope in a trend-reduced Markoff time series. Two smoothing formulas, each extending over $2k + 1$ terms, were considered: (i) a moving average of 3 separated terms, and (ii) an equally weighted moving average. Tables 1 and 2 give results to 2D for $k = 1, 2, 3, 5, 10, \infty$ and ρ_ϵ, $\rho_x = 0(.1).9$ which allows one to make an optimum choice of k if the values of other parameters are known.

Tables 5 and 6 give, for the same values of k and the two parameters noted above, correction factors to 2D for converting a classical estimate of this same regression slope into an estimate which is adjusted for autocorrelations in the series.

Tables 3 and 4 give, for the above smoothing formulas, the first serial correlation to 2D of the reduced series for the same values of k in terms of the Markoff parameter $\rho = 0(.1).9$. Used inversely, these tables may be used to obtain estimates of ρ from the observed serial correlations.

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29[K].—E. S. PAGE, "A test for a change in a parameter occurring at an unknown point," *Biometrika*, v. 42, 1955, p. 523–527.

The statistic $m = \max_{0 \le i \le n} \{S_r - \min_{0 \le i < r} S_i\}$, where $S_r = \sum_{j=1}^{r} (x_j - \theta)$, $S_0 = 0$, is suggested to test the hypothesis that the observations x_i, $i = 1, \cdots, k$ (in order of observation) have mean value θ and the observations x_i, $i = k + 1, \cdots, n$ have mean value $\theta' > \theta$. For the special case $y_i = \text{sgn}\,(x_i - \theta)$ and x_i symmetrically distributed, 5% and 1% points are derived for values of n ranging from 21 to 185 (Table 1). The power of this test is compared to that of the usual sign test (values to 3D) for $k = 0$ and $p = \Pr[x_i - \theta > 0 | E(x_i) = \theta' > \theta] = 0.5(.05).8$, and for $p = .75$ and $k = 0(10)50$ (Tables 2 and 3).

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30[K].—R. L. STORER & W. R. DAVISON, "Simplified procedures for sampling inspection by variables," *Industrial Quality Control*, v. 12, no. 1, 1955, p. 15–18.

The authors present master tables for sampling inspection by inspection based on the average range ($\bar{R}$) and sample standard deviation (V). The procedure based on the average range is: (1) accept the lot if $\bar{\bar{X}} + k\bar{R} \le U$ when there is a one-sided upper specification limit (U) given; (2) accept the lot if $\bar{\bar{X}} - k\bar{R} \ge L$ when there is a one-sided lower specification limit (L) given; and (3) accept the lot if $\bar{\bar{X}} + k\bar{R} \le U$ and $\bar{\bar{X}} - k\bar{R} \ge L$ and $\bar{R} \le$ maximum allowable average range ($MA\bar{R}$) where $MA\bar{R} = F(U - L)$ when there is a two-sided specification limit given. The authors give values of k to 2D and F to 3D for sampling plans indexed according to acceptable quality levels of .065, .1, .15, .25, .4, .65, 1, 1.5, 2.5, 4, and 6.5 (in percent) and sample sizes of 5, 10, 15, 20, 25, 30, 35, 40, 50, 70, 130, and 200.

The procedure based on the sample standard deviation is the same as the above with $\bar{R}$ replaced by V. The authors give values of k to 2D and F to 3D for the same parameters as given above.

The authors state that the plans based on the average range and sample standard deviation give essentially equal protection for a given sample size in the sense of having the same operating characteristic curves. However, no operating characteristic curves are presented in the article but appear in the stated references.

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31[K].—G. J. LIEBERMAN & HERBERT SOLOMON, "Multi-level continuous sampling plans," *Annals Math. Stat.*, v. 26, 1955, p. 686–704.

We quote from page 687 of the article:

"0) At the outset inspect 100 percent consecutively as produced and continue such inspection until i units in succession are found clear of defects.

"1) When i units in succession are found clear of defects, discontinue 100 percent inspection and inspect only a fraction f of the units (i.e., one out of every

$1/f$ where $1/f$ is an integer). If the next i inspected units are non-defective, proceed to the next level; if a defective occurs, revert immediately to 100 percent inspection.

"2) When at rate f, i inspected units are found clear of defects, discontinue sampling at rate f and proceed to sampling at rate f^2. If the next i inspected units are non-defective, proceed to the next level; if a defective occurs, revert immediately to sampling at rate f.

"3) When at rate f^2, i inspected units are found clear of defects, discontinue sampling at rate of f^2 and proceed to sampling at rate f^3. If the next i inspected units are non-defective, proceed to the next level; if a defective occurs, revert immediately to sampling at rate f^2.

. . .

"$k-1$) When at rate f^{k-2}, i inspected units are found clear of defects, discontinue sampling at rate f^{k-2} and proceed to sampling at rate f^{k-1}. If the next i inspected units are non-defective, proceed to the next level; if a defective occurs, revert immediately to sampling at rate f^{k-2}.

"k) When at rate of f^{k-1}, i inspected units are found clear of defects, discontinue sampling at rate f^{k-1} and proceed to sampling at rate f^k. If a defective occurs, revert immediately to sampling at rate f^{k-1}, otherwise, continue sampling at rate f^k.

"Whenever sampling is in operation, one item should be selected at random from each segment of $1/f^j$ $(j = 0, 1, 2, \cdots, k)$ production items. During both sampling inspection and 100 percent inspection all defective items found should either be corrected or replaced with good items."

For $k = 1$, the Lieberman-Solomon plan reduces to Dodge's 1943 plan [1].

Charts show the relationship of f to i for the average outgoing quality limit (*AOQL*) in percents .01, .02, .03, .05, .1, .2, .3, .5, 1(1)6, 8, 10, for $k = 1, 2, \infty$.

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1. H. F. DODGE, "Skip-lot sampling plan," *Industrial Quality Control*, v. XI, no. 5, 1955, p. 3–5. [*MTAC*, v. 10, 1956, RMT 19, p. 47.]

32[K].—H. F. DODGE, "Chain sampling inspection plan," *Industrial Quality Control*, v. 11, No. 4, 1955, p. 10–13.

A chain sampling plan introduces a somewhat different consideration in sampling inspection. The plan overcomes—to a degree—the shortcomings of a sampling plan involving a single small sample with an acceptance number, $c = 0$.

The procedure of the plan is as follows: (a) For each lot, select a sample of n units and test each unit for conformance to a specified requirement; (b) The acceptance number of defects is $c = 0$; except that $c = 1$ if no defects are found in the immediately preceding "chain" of i samples of size n. $(i = 1, 2, 3, \cdots.)$ That is, a lot is accepted if no defects are found in its sample of n units. A lot is rejected if two or more defects are found in its sample. But if one defect is found, the lot can still be accepted if the last defect was far enough back in the history of the product, as determined by the choice of i.

The characteristic curves for four sets of chain sampling plans designated by ChSP-1 are given. The sample sizes are $n = 4, 5, 6$ and 10 with values of $i = 1(1)5$.

A formula is presented to be used for drawing up curves for arbitrarily chosen values of n and i.

Because of the cumulative aspect of chain sampling, it is obvious that there exist conditions under which the plan is best applied. The author clearly states these conditions which have been assumed when generating the theory fundamental to chain sampling.

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33[K, L].—NBS Applied Mathematics Series, No. 41, *Tables of the Error Function and its Derivative* $H(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\alpha^2} d\alpha$ *and* $H'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$, U. S. Gov. Printing Office, Washington, D. C., 1954, xi + 302 p., 26 cm. Price $3.25.

Here are listed $H'(x)$ and $H(x)$, $x = 0(.0001)1(.001)5.6$(various), 15D and $H'(x)$ and $1 - H(x)$, $x = 4(.01)10$, 8S, where $H(x)$ is the error function, $H(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\alpha^2} d\alpha$.

This is a reissue of "Tables of probability functions," v. 1, 1941, Mathematical Table 8, prepared by the Mathematical Tables Projects of the Federal Works Agency, Works Projects Administration (see RMT **91**, *MTAC*, v. 1, 1943, p. 48–51). The only change has been the correction of two minor misprints, and one major misprint; the correct value at $x = 1.742$ is $2\pi^{-\frac{1}{2}} e^{-x^2} = .05427\ 01046\ 62097$.

The bibliography of tables in this volume was not supposed to be complete, but it is regrettable that this new edition included no reference to [1] (see RMT **1034**, *MTAC*, v. 6, 1952, p. 232). Incidentally in [1] the phrase 'error function' is used to denote a quantity which is one-half less than the normal probability integral—a notation not in conformance with the NBS notation or with Fletcher, Miller, and Rosenhead's, *An Index of Mathematical Tables*.

Topography of the new edition remains adequate.

C. B. T.

1. Harvard University, Computation Laboratory, *Annals*, v. 23, *Tables of the Error Function and its First Twenty Derivatives*, Cambridge, Mass., Harvard University Press, 1952.

34[L].—S. Johnston, *Tables of Sievert's Integral*, Manchester and Newcastle upon Tyne, England, 1955, 23 × 32 cm. (oblong), 9 pages. Copies may be purchased from the Royal Soc. Depository of Unpublished Mathematical Tables, Burlington House, London W. 1, and also from the Hospital Physicists Association, Diagrams and Data Scheme, Mount Vernon Hospital, Northwood, Middlesex, England. Two copies have been deposited in UMT File.

These tables are, apart from details of arrangement, the same as the tables computed in 1948 by S. Johnston, 81 Fountain Street, Manchester, 2, England, in response to queries by R. D. Evans in *MTAC*, and briefly described in UMT **103** (*MTAC*, v. 4, 1950, p. 163). It may be recalled that Sievert's integral

$$\int_0^\pi e^{-A \sec \theta} d\theta$$

is used in radio-therapy (see *MTAC*, v. 2, 1946, p. 196). The copies now reviewed were made in 1955 by the Radiotherapy Department, Royal Victoria Infirmary, Newcastle upon Tyne, England, although this fact is not stated on them. The copying was done from a typed manuscript by photo-lithography, some of the headings being printed in at the same time. Values of the integral are given to about 5S, without differences, for $A = 0(.5)10$, $x = 0(1°)90°$. More precisely, the number of decimals depends upon the value of A, as follows:

5D	$A = 0.0$ to 1.5
6D	$A = 2.0$ to 4.0
7D	$A = 4.5$ to 6.0
8D	$A = 6.5$ to 8.0
9D	$A = 8.5$ to 10.0

In the version described in UMT **103**, explicit tabulation was not made beyond a value of x after which the integral remains unchanged to 5S; but in the present version all values are explicitly tabulated. It may be added that the National Bureau of Standards is producing more extended tables and that S. Johnston has computed many thousands of values of related integrals.

A. F.

35[L, S].—C. E. FRÖBERG, "Numerical treatment of Coulomb wave functions," *Rev. Mod. Phys.*, v. 27, 1955, p. 399–411.

Coulomb wave functions are particular forms of confluent hypergeometric functions which are of importance in nuclear physics. They are the solutions of the differential equation

$$\frac{d^2y}{d\rho^2} + \left\{1 - \frac{2\eta}{\rho} - \frac{L(L+1)}{\rho^2}\right\} y = 0,$$

in which L (an integer) and η are parameters. The most extensive tables of solutions [1] cover the ranges $0 \leq L \leq 21$, $-6 \leq \eta \leq 6$, $0 \leq \rho \leq 5$. The purpose of the present paper is to give formulae from which isolated values lying outside the range of these tables may be computed to an accuracy of at least five or six significant figures.

The formulae are drawn from the many scattered papers on the subject and include recurrence relations in the L-direction, integral representations, ascending series in ρ, asymptotic series in $1/\rho$, expansions at the transition point $\rho = 2\eta$, and some expansions in terms of Bessel and Airy functions. Useful charts are included which indicate recommended methods in various regions.

The disadvantage of most of the expansions given is that they are useful in only relatively small regions. The theory of the asymptotic solution of differential equations containing a parameter can be applied to the Coulomb wave equation to determine expansions for large η, L in terms of Bessel, Airy or exponential functions, which are uniform with respect to *unrestricted* ρ, unlike those given in the paper. Leading terms of such expansions have been given in more recent

publications [2, 3, 4] and higher terms can be obtained, for example, by application of the theory of the reviewer [5, 6].

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1. NBS Applied Mathematics Series, No. 17, *Tables of Coulomb Wave Functions*, v. 1, U. S. Gov. Printing Office, Washington, D. C., 1952.
2. A. ERDÉLYI, M. KENNEDY, & J. L. MCGREGOR, "Asymptotic forms of Coulomb wave functions I," California Institute of Technology: Tech. Report No. 4 NRO43–121, 1955.
3. A. ERDÉLYI & C. A. SWANSON, "Asymptotic forms of Coulomb wave functions II," California Institute of Technology: Tech. Report No. 5 NRO43–121, 1955.
4. N. D. KAZARINOFF, "Asymptotic expansions for the Whittaker functions of large complex order *m*," Amer. Math. Soc., *Trans.*, v. 78, 1955, p. 305–328.
5. F. W. J. OLVER, "The asymptotic solution of linear differential equations of the second order for large values of a parameter," Roy. Soc., *Phil. Trans.*, Ser. A, v. 247, 1954, p. 307–327.
6. F. W. J. OLVER, "The asymptotic solution of linear differential equations of the second order in a domain containing one transition point," Roy. Soc., *Phil. Trans.*, Ser. A, v. 249, 1956, p. 65–97.

36[L].—E. E. ALLEN, "Polynomial approximations to some modified Bessel functions," *MTAC*, v. 10, 1956, p. 162–164.

Polynomial approximations for the following functions: $I_0(x)$, $-3.75 \leq x \leq 3.75$, 8D; $I_1(x)/x$, $-3.75 \leq x \leq 3.75$, 8D; $I_0(x)x^{\frac{1}{2}}e^{-x}$, $3.75 \leq x < \infty$, 7D; $I_1(x)x^{\frac{1}{2}}e^{-x}$, $3.75 \leq x < \infty$, 7D; $K_0(x) + \log_e(.5x)I_0(x)$, $0 < x \leq 2$, 7D; $[K_1(x) - \log_e(.5x)I_1(x)]x$, $0 < x \leq 2$, 7D; $K_0(x)x^{\frac{1}{2}}e^x$, $2 \leq x < \infty$, 7D; $K_1(x)x^{\frac{1}{2}}e^x 2 \leq x < \infty$, 7D.

C. B. T.

37[L, S].—D. K. C. MACDONALD & LOIS T. TOWLE, "Integrals of interest in metallic conductivity," *Canadian J. of Physics*, v. 34, 1956, p. 418–19.

This note gives a table of $J_r(x)$,

$$J_r(x) = \int_0^x \frac{z^r dz}{(e^z - 1)(1 - e^{-z})}$$

for $r = 2, 3, 4$, and 6; $x = 0.1, 0.25, 0.5, 1.0, 1.2, 1.5, 2(1)6, 8, 10, 13, 20$, and ∞; for 4–6S.

This table contains the following errors, together with some one or two digit errors in the last significant figure:

x	$J_4(x)$ *is*	*should be*
2	2.201*6*	2.2011
3	5.96*32*	5.9648
4	10.7*293*	10.7319
5	15.3*671*	15.3598
6	19.12*10*	19.1230
8	23.587*4*	23.5840
10	25.28*12*	25.2737
13	25.*9273*	25.8860
20	25.*9639*	25.9754

x	$J_6(x)$ *is*	*should be*
3	29.*69*	29.58
6	295.*40*	295.43
8	507.*00*	506.81
10	63*9.21*	638.78

The computation on SEAC was done using Simpson's Rule and integrating from zero. The end values from SEAC were checked by the asymptotic formula developed below.

As pointed out in the note $J_4(\infty) = 4\pi^4/15$ and $J_6(\infty) = 16\pi^6/21$, from this we see

$$J_4(x) = \frac{4\pi^4}{15} - I_4(x)$$

where

$$I_4(x) = \int_x^\infty z^4 e^{-z} \left\{1 + \frac{e^{-z}}{1 - e^{-z}}\right\}^2 dz$$

$$I_4(x) = \int_x^\infty z^4 e^{-z} dz + E_4(x)$$

$$I_4(x) = e^{-x}\{x^4 + 4x^3 + 12x^2 + 24x + 24\} + E_4(x)$$

where

$$|E_4(x)| < (2e^{-x} + e^{-2x})I_4(x) \quad x > 0.$$

From the above formula for $J_4(x)$ and a similar formula for $J_6(x)$, $J_4(20)$ and $J_6(20)$ computed by the integration on SEAC were found to be accurate to within one unit in the eight-figure. The corrections listed for $J_4(x)$, $x = 10, 13, 20$, and $J_6(x)$, $x = 10$, were verified by the asymptotic formulas as well as the computation on SEAC.

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38[S, X].—R. A. SPARKS, R. J. PROSEN, F. H. KRUSE, & K. N. TRUEBLOOD, "Crystallographic calculations on the high-speed digital computer SWAC," *Acta Cryst.*, v. 9, 1956, p. 350–358.

The paper contains descriptions and flow diagrams for structure-factor calculations, Fourier summations, differential Fourier summations, and a least-squares refinement used in crystallographic calculations on SWAC. The codes are constructed to accept any space group and to be suitable for crystals containing up to two hundred different anisotropic atoms or one thousand isotropic atoms. The discussion of the differential Fourier summations is not as complete as that of the other parts of the calculation.

The authors note that, in addition to these major routines, they have complied several short routines for special purposes. These include correction factors for Weissenberg intensity data, calculation of interatomic distances and angles, and location of maxima on Fourier syntheses.

So far as the reviewer knows the largest crystal treated by the method so far was the vitamin B_{12} fragment [1]. The routines have also been used to analyze about thirty other crystals.

C. B. T.

1. D. C. HODGKIN, J. PICKWORTH, J. H. ROBERTSON, J. G. WHITE, K. N. TRUEBLOOD, & R. PROSEN, "The crystal structure of the hexacarboxylic acid derived from B_{12} and the molecular structure of the vitamin," *Nature*, v. 176, 1955, p. 325–328.

39[T, Z].—T. R. NORTON & A. OPLER, *A Manual for Coding Organic Compounds for Use with a Mechanized Searching System*, The Dow Chemical Co., Pittsburg, Calif., 1953 (Revised 1956), 56 p., 28 cm.;

A. OPLER & T. R. NORTON, *A Manual for Programming Computers for Use with a Mechanized System for Searching Organic Compounds*, The Dow Chemical Co., Pittsburg, Calif., 1956, 23 p., 28 cm.; and "New speed to structural searches," *Chem. and Eng. NEWS*, v. 34, 1956, p. 2812–2816.

One of the many applications of modern computers which particularly interests the chemist is the mechanization of searching the ponderous accumulation of literature in his field. It will be some time before the ultimate electronic library is achieved, but significant steps are being made and the work of Opler and Norton is one of them. They have devised a technique for "For Coding Organic Compounds" and put it to practice with two recent computers, the DATATRON and the I.B.M. 701. Their purpose was to rapidly search the thousands of known organic chemical compounds for correlating physical, chemical, or biological properties with structure. The results of their efforts to date are summarized in an article in Chemical and Engineering News [1] and the techniques are elaborated in the two manuals available from The Dow Chemical Company. The article, "New Speed to Structural Searches," provides a good introduction to and review of the problems and how they are being solved. The heart of the rapid searching technique is the coding system employed to translate the organic structure into numerical form. While a number of systems have been devised, the system of Opler and Norton represents a relatively complete reduction of organic structure notation to numerical form. This, in effect, makes the coding of searching and related routines a straightforward task for most computers.

The coding system detailed in the first manual is built on a sequence of seven-digit numbers. There will be as many of these numbers in a sequence as there are chemical groups in the compound being coded. The system is designed to handle most types of organic compounds now known and is sufficiently flexible to accommodate some of the special classes of compounds currently omitted. The coding manual is clear, concise, and adequately stocked with examples. However, in the words of the authors, "The ultimate value of this convention (of coding) can only be shown by operating experience."

The manual for programming the computer gives a general outline of the type of program which the authors prepared to process the coded compounds and to search the file. Due to the various modes of programming now employed in current computers the manual was written very generally. It is difficult to decide how general or specific to make such a manual. From the point of view of those chemists who have done some coding, the compound notation system would

become more acceptable if a few extra pages of detailed example were appended to the coding manual. Aside from that small point the manual very well outlines the structure of the programs.

Opler and Norton are to be commended for their fine work in this and other applications of computers to chemical problems and The Dow Chemical Company should also be commended for making this information generally available.

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40[T, Z].—THOMAS B. DREW & JOHN W. HOOPES, JR., editors, *Advances in Chemical Engineering*, Academic Press, Inc., 1956, x + 448 p. Price $10.00.

It is most amazing that the use of computers has not been appreciated by the chemical industry, in contrast, say, with the aircraft industry which has been using them for decades. Nobody would build an airplane without the aid of a computer, yet plants are still built with a slide rule. It is a pity that a book of this title should not have a chapter on the use of computers in chemical engineering, for there are innumerable places where automatic computing methods would advance engineering analysis very profitably. One can quote, for example, where the use of the equation of state instead of the perfect gas law has revolutionized certain engineering designs. Perhaps it is the wide variety of possible applications which has failed to arouse the interest of the chemical engineer.

This book has one chapter on computers, written by Robert Schrage, devoted to their application to the control and planning of manufacturing operations. One gets the impression that this is an apology for computers, and that the applications have been very spotty. No large industry relies on a computer for control of its operation.

With this background it is not surprising that this chapter is very general in its discussion of computing equipment, and serves merely as a source of references. The manufacturer or engineer who wants guidance in getting started will find no facts about specific machines in regard to capability, speed, reliability, etc. Not one of the large computers is mentioned by name or number, and it is to be hoped that the chemical industry is not, at this late date, going to be founded on the IBM, CPC, which is the only machine mentioned.

Four sections of the chapter are devoted to more or less detailed descriptions of certain applications. No information is given regarding the time or cost of carrying out these applications, and whether they were economically a success. Indeed, most of the examples were merely exploratory. The author misses an important point, if he is attempting to stimulate interest in computers, among chemical engineers, in his discussion of blending. The model given is so naive that it has no practical application. There are "technological complications." Why not point out that an automatic computer can permit the introductions of great complexity, to make a model realistic, far beyond the capabilities of manual methods?

Most of the examples, Monte Carlo method, factorial designs, method of steepest descents, and linear programming are more truly problems in operations

research, and are more fully treated in books on that subject. This chapter is not specific as to how one actually puts these problems on a computer.

For general review of what is being done with computers in chemical engineering, the chapter in this book is readable, concise, and clear. The young engineer will not find much specific or stimulating. The computer will find no unusual applications, but the account of the computing approach to the method of steepest descents in sequential (statistical) design, and optimization in the presence of restrictions, are quite readable introductions.

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41[W, X, Z].—M. V. WILKES, *Automatic Digital Computers*, John Wiley & Sons, Inc., New York, 1956, x + 305 p., 22 cm. Price $7.00.

In the rapidly changing field of automatic digital computers an occasional summing up and surveying of the prospect is necessary. This book introduces the reader to the principles underlying the design and use of present-day computers. This is done by examples of the construction, design, and application of existing machines. A general treatment of logical design and programming is coupled with a description of many types of storage mediums and arithmetic units. Numerical analysis is not treated.

The subject is introduced by a historical chapter on the development of computers, but even at this stage the author introduces logical elements and describes flipflops, gates, and counters in some detail as applied in the circuits of ENIAC and EDVAC. The principles of logical design he derives from a detailed study of EDSAC, the serial machine at Cambridge. Indeed, throughout the book, his emphasis in each instance is on the serial machine, followed by a discussion of the corresponding parallel logic. This is not a fault, but undoubtedly stems from his experience with serial computers.

A chapter is devoted to relay computers. The author admits that these have probably reached their ultimate in development due to the physical limitations of the relays themselves, but feels that a study of relay computing circuits will aid in the general field of computing machine design. Here again, logical principles are demonstrated by their application in existing machines: the Bell Telephone Laboratory computers and the Harvard computers.

Storage forms discussed in detail include: ultrasonic, both solid and liquid, electrostatic, and magnetic (wire, tape, drum, and core). Electronic switching circuits are treated systematically in a separate chapter, although descriptions of computer circuits occur at many places throughout the text.

The treatment of programming is quite complete, with examples again derived from EDSAC material. It includes machine language coding, master and interpretive routines, subroutines, and symbolic, or relative address coding. It is a thorough general discussion of basic coding practice and present trends toward more elaborate programs. A final chapter discusses the problems of choosing a design for a computer and the organization of a computing center. The author's opinions are clearly derived from extensive experience with both computers and operating personnel. He has devoted a few pages to an Appendix, entitled,

"Machinery and Intelligence," which settles, as far as he is concerned, the question as to whether machines can "think."

The book is not intended as a textbook. However, with the aid of the annotated bibliography to provide additional subject matter in connection with certain details of machine construction or program arrangement, as needed, it might prove acceptable as such in an introductory course on digital computers.

One criticism may be launched against the discussion of storage devices. Punched paper tape, punched cards, and various forms of printers and photographic storage devices are discussed under the heading of logical design. They could well have been treated in more detail in the chapter on storage devices, or separately, as input and output devices.

Another difficulty lies in the heavy emphasis on EDSAC, particularly in the chapter on program construction. The problem of having to specialize the subject matter in favor of one computer instead of another is one which any author in this field must face. The reader, familiar though he may be with some computer, will generally be forced to learn the language of another. The choice of machine for illustration will be made, then, dependent on the author's experience, as in this case.

But these are minor criticisms of what is, in fact, a very complete account of the subject. For the layman, and for most mathematicians and engineers, this book presents a lucid, readable summary of computer design and operating principles. An adequate number of diagrams supports the descriptive material.

The list of chapters follows: 1. The Development of Automatic Digital Computers, 2. The Principles of Logical Design, 3. The Principles of Programme Construction, 4. Relay Computers, 5. Storage, 6. Electronic Switching and Computing Circuits, 7. The Design and Operation of Digital Computers, Appendix: Machinery and Intelligence, Annotated Bibliography, and Index.

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42[X].—NBS Applied Mathematics Series, No. 12, *Monte Carlo Method. Proceedings of a Symposium held on June 29, 30, and July 1, 1949*, in Los Angeles, California, under the sponsorship of the RAND Corporation, and the National Bureau of Standards, with the cooperation of the Oak Ridge National Laboratory (A. S. Householder, editor). U. S. Gov. Printing Office, Washington, D. C., 1951, vii + 42 p, 26.0 cm. Price 30¢.

Contents: Preface by J. H. Curtiss; Foreword by A. S. Householder; "Showers produced by low-energy electrons and photons," by R. R. Wilson; "An alignment chart for Monte Carlo solution of the transport problem," by B. I. Spinrad, G. H. Goertzel, and W. S. Snyder; "Neutron age calculations in water graphite, and tissue," by A. S. Householder; "Methods of probabilities in chains applied to particle transmission through matter," by W. C. DeMarcus and Lewis Nelson; "Stochastic methods in statistical mechanics," by G. W. King; "Report on a Monte Carlo calculation performed with the ENIAC," by Maria Mayer; "Calculation of shielding properties of water for high energy neutrons," by P. C. Hammer; "A Monte Carlo technique for estimating particle attenuation in bulk

matter," by B. A. Shoor, L. Nelson, W. DeMarcus, and R. L. Echols; "Estimation of particle transmission by random sampling," by H. Kahn and T. E. Harris; "History of RAND'S random digits—Summary," by G. W. Brown; "The mid-square method of generating digits," by P. C. Hammer; "Generation and testing of random digits at the National Bureau of Standards, Los Angeles," by G. E. Forsythe; "Various techniques used in connection with random digits," by J. von Neumann; "Round table discussion" (chaired by J. W. Tukey); summary by H. H. Germond.

Four of the papers deal with random or pseudo-random digits and most of the others deal with the progress of elementary particles through matter.

Although the symposium was held more than seven years ago these proceedings give a good impression of the ideas and uses of the Monte Carlo method. The article by H. Kahn and T. E. Harris (pages 27–30) is especially informative, and describes three techniques for reducing sampling variance. These techniques are: (i) the splitting technique, in which important particles are conceptually split into two; (ii) importance sampling in which the probability distributions of the original model are altered so as to spend more of the sampling effort in important regions of phase space; and (iii) the technique of making a Monte Carlo method as little like Monte Carlo as possible. "That part of the problem that is hard to do analytically is done by sampling, and that part that is easy to do analytically, is so done." J. von Neumann's influence in all these techniques is acknowledged.

The paper on the generation and testing of random digits at Los Angeles contains a slip in the application of the serial test. This slip is one of principle and had been made before and will probably be made again. The principle has since been corrected: see the reviewer's paper [1], and C. B. Tompkins's review of RAND's Million Random Digits [*MTAC*, v. 10, Rev. 11, 1956, p. 40 and 42].

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1. I. J. Good, "The serial test for sampling numbers and other tests for randomness," Camb. Phil. Soc., *Proc.*, v. 49, 1953, p. 276–284.

43[X].—H. A. Meyer, Editor, *Symposium on Monte Carlo Methods*, held at the University of Florida, March 16 and 17, 1954. John Wiley & Sons, Inc., New York, 1956, p. xvi + 382, 28.5 cm. Price $7.50.

Contents: "An introductory note," by A. W. Marshall; "Generation of pseudo-random numbers," by O. Taussky and J. Todd; "Phase shifts—middle squares—wave equation," by N. Metropolis; "A general theory of stochastic estimates of the Neumann series for the solution of certain Fredholm integral equations," by G. E. Albert; "Neighbor sets for random walks and difference equations," by T. S. Motzkin; "Monte Carlo computations," by Nancy M. Dismuke; "Applications of Monte Carlo methods to tactical games," by S. Ulam; "Conditional Monte Carlo for normal samples," by H. F. Trotter and J. W. Tukey; "Monte Carlo techniques in a complex problem about normal samples," by H. J. Arnold, B. D. Bucher, H. F. Trotter, and J. W. Tukey; "An application of the Monte Carlo method to a problem in gamma ray diffusion," by M. J. Berger; "Stochastic

calculations of gamma ray diffusion," by L. A. Beach and R. B. Theus; "Application of multiple-stage sampling procedures to Monte Carlo problems," by A. W. Marshall; "Questionable usefulness of variance for measuring estimate accuracy in Monte Carlo importance-sampling problems," by J. E. Walsh; "Experimental determination of eigenvalues and dynamic influence-coefficients for complex structures such as airplanes," by C. W. Vickery; "Use of different Monte Carlo sampling techniques," by H. Kahn; "A theoretical comparison of the efficiencies of two classical methods and a Monte Carlo method for computing one component of the solution of a set of linear algebraic equations," by J. H. Curtiss; "A description of the generation and testing of a set of random normal deviates," by E. J. Lytle, Jr.; "Machine sampling from given probability distributions," by J. W. Butler; "A Monte Carlo technique for obtaining tests and confidence intervals for insurance mortality rates," by J. E. Walsh; "Experiments and models for the Monte Carlo method," by A. Walther; Bibliography. The bibliography extends from page 284 to 370 and contains many abstracts.

The expression, "Monte Carlo method," has changed its meaning somewhat during the last eight years. At first it described a method of approximating the numerical solutions of ordinary mathematical problems by probabilistic methods. A one-person game or artificial sampling experiment is performed in which the expected score is the required answer and the observed score is an estimate of it. An example is to go for a random walk in order to invert a matrix, or to throw needles on a grating in order to evaluate π. Judging by most of the applications to date the name now describes a similar method for attacking problems that for the most part originate in a probabilistic setting. The methods now use several techniques for the reduction of the variance of the answer. Some of these techniques, such as systematic and stratified sampling, have long been known to statisticians; others are at least partly new. Often the technique can be expressed by saying that the probability model used in the Monte Carlo method is not quite the same as the model from which the model arose. A useful summary of six of these techniques is given in the paper by H. Kahn (p. 146–190). This paper is very readable, and so are most of those following it in the book. But most of the preceding ones are almost in note form and have signs of hurried preparation.

The introductory article contains the following passage: "The Trotter and Tukey paper, with its companion applied paper ··· [by Arnold, Bucher, Trotter, and Tukey], is probably the most exciting paper in the volume. First, the authors introduce a new variance reducing technique (conditional Monte Carlo) and second, in the applied paper they show that it works very well. It is heartening to see mathematical statisticians really becoming interested in Monte Carlo techniques and using them on their own problems. This paper deserves close study (and needs it) to get the conditional Monte Carlo trick straight. It seems to be more complicated than, for example, the methods discussed by Kahn."

Curtiss's paper gives a detached comparison of the Monte Carlo method, with classical methods, for solving a set of linear algebraic equations. The Monte Carlo method does not compare favourably on the whole. [Cf. J. TODD, "Experiments on the inversion of a 16 × 16 matrix," NBS Applied Mathematics Series No. 29, 1953, p. 113–115.] On pages 192–3 Curtiss gets close to noticing that Hausdorff summation is in effect a Monte Carlo process.

Four of the papers relate to the generation of pseudo-random numbers and other random variables. One of these methods, the mid-square method, exemplifies how the Monte Carlo method may lead indirectly to new developments in mathematics and technology. For eight-digit binary numbers the mid-square method may be defined by the iteration $u_{n+1} = [u_n^2/16] \pmod{256}$. This example is illustrated (page 31) by means of an oriented linear graph with one vertex for each of the 256 numbers (except that two vertices seem to be missing). As pointed out by Metropolis and Ulam (*Amer. Math. Monthly*, v. 50, 1953, p. 252–253) the frequencies of the numbers of predecessors of the vertices seem to obey a Poisson distribution. Thus the oriented linear graph can be regarded from at least one point of view as a pseudo-random network. This fact may be of interest in mathematical biophysics or in robotology for constructing mathematical models of the central nervous system. If robots are mass-produced it would be advisable for their brains to be pseudo-random networks, and not random ones, so that the study of the behaviour of prototypes would be a reliable guide to the behaviour of marketed models. This advantage of pseudo-random networks over random ones is analogous to the checking advantage that pseudo-random numbers have in ordinary Monte Carlo.

On page 33 there is a table of the number of trees in the oriented linear graphs corresponding to the mid-square method using a binary base and 6(2)20 digits. On page 34 there is a table of the number of trees when there are two and four digits and the base is 2(1)16; six digits, base = 2(1)8; eight digits, base = 2(1); and ten digits, base = 2, 3.

On page 224 there is a table, by Curtiss, of the favourable sizes of matrices for three methods of inversion, namely the Gauss elimination method, the linear iterative method, and a Monte Carlo method. Results are given for the two cases when the maximum sum of the moduli of the elements of a row of the original matrix is 0.5 and 0.9, and for various numbers of significant decimal places in the elements of the inverse.

On pages 239 and 240 there are tables, by Lytle, of the frequencies of the ranges 1.1(0.1)5.9 (sample size 10) and 2.2(0.1)6.4 (sample size 25) for samples of random normal deviates. These frequencies were obtained by a crude Monte Carlo method or artificial sampling experiment. The total frequencies for the two experiments were respectively 2500 and 1000.

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44[X, Z].—*Numerical Analysis*, v. 6, *Proceedings of the Sixth Symposium in Applied Mathematics of the American Mathematical Society*, held in Santa Monica, Calif., Aug. 26–28, 1953, edited by John W. Curtiss (Editorial Committee: R. V. Churchill, Erich Reissner, A. H. Taub), McGraw-Hill Book Co., New York, 1956, vi + 303 p. Price $9.75.

This book contains papers presented at the Sixth Symposium in Applied Mathematics of the American Mathematical Society. The subject of this Symposium was Numerical Analysis, the Symposium being held at Santa Monica City

College, Santa Monica, California, August 26 to 28, 1953. The National Bureau of Standards was a cosponsor, John H. Curtiss, the editor.

With its 19 papers the book gives a vivid survey of large parts of modern applied mathematics. It is a demonstration of the rapid changes having taken place in the field of applied mathematics during the past fifteen years. All papers refer more or less closely to computations using modern high-speed computing devices. The methods of numerical analysis are evaluated in respect to their applicability to High Speed Computers. Several papers deal with the problem of the accuracy of the results obtained, a question which is of special importance for more extensive computations. We can here of course only give a short survey of the comprehensive contents.

Systems of linear equations—a very important subject—are dealt with in several contributions. David Young gives a summary of the development for iteration methods in particular and proposes two improvements. The increase in speed of convergence theoretically to be expected is confirmed by computations on ORDVAC. Joseph W. Fischbach brings some applications of the gradient method and applies the methods of steepest descent to ordinary differential equations, Magnus R. Hestenes uses the cg method (conjugate gradient method) for systems of infinitely many linear equations in Hilbert space, and Paul Rosenbloom gives a comprehensive theory covering partial differential equations, too. Another publication about parabolic equations will follow.

A number of contributions deal with approximation theory. Cecil Hastings, Jr., Jeanne Hayward, and James P. Wong, Jr. report about examinations lasting several years aiming to approximate given functions not only by polynomials but by rational or more general parametric forms (containing, for example, roots) for easier feeding of the functions into computing machines. J. L. Walsh deals with Chebychev's polynomial approximation and Arthur Sard with approximations for derivatives and integrals. Sard includes functions of two independent variables, where the residues are constructed as linear functionals in Banach space and estimations for them thus obtained.

For some time now problems of number theory have been treated on computing machines, e.g., combinatorical problems by R. H. Bruck, and problems of algebraic number theory by Olga Taussky. Progress in the theory of algebraic number fields in particular is hindered by the greatly increased difficulties of numerical examples. Olga Taussky deals with factorization of rational primes, ideal classes, and a problem of Hilbert. Emma Lehmer gives three classes of problems suitable for computing machines: special problems, testing of hypotheses, and research problems. T. S. Motzkin deals with the interesting assignment problems, with normalization, approximate solution, vertex approach and face approach methods, equidistribution problem and related problems. C. Tompkins examines problems whose variables are permutations and reports about the treatment of problems with SWAC. His report includes systematic machine generation of permutations, embedding of permutations in continuous spaces, and the problem of economic computing. Richard Bellman deals with the theory of dynamic programming and obtains a functional equation the approximate solution of which is obtained by successive approximation.

Of great importance is further research in differential equations and related

fields. Stefan Bergman points to the kernel functions for solving boundary value problems. They are useful when more then a single boundary value problem has to be solved.

With their aid one can estimate how the solution of a linear partial differential equation changes with changes of the domain for which it is considered, of the coefficients of the differential equation or of the boundary values. The methods are well suited for High Speed Computers. The paper includes numerous very instructive examples. S. P. Frankel reports about the stability of the finite difference method for partial differential equations in special cases. D. R. Clutterham and A. H. Tabu examine the shock configuration in Mach reflection; the method worked out by them stood the test on ILLIAC; two pages of tables give results.

S. E. Warschawski collects different methods of approximative conformal mappings, integral equation methods, mapping of nearly circular domains according to M. S. Friberg, the Rayleigh-Ritz-Method, and the "perturbation method." These methods are theoretically examined, in some cases estimations of error are given and tested numerically for ellipses.

Helmut Wielandt gives an important methodical estimate of the eigenvalues for second kind integral equations with symmetric kernel. An error estimate is carried out for approximations obtained by performing the integration approximately by a formula of numerical quadrature. Special integration formulas are examined and numerical examples given. Wolfgang Wasow deals with the problem of asymptotic transportations of certain distributions into the normal distribution. His examinations include Student's and the χ^2 distribution and possibly other distributions of interest.

Thus the volume presents very interesting aspects of a number of modern research areas.

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45[X, Z].—CLARENCE L. JOHNSON, *Analog Computer Techniques*, McGraw-Hill Book Co., New York, 1956, 280 p. Price $6.00.

This book was written as a textbook for a course in the operation of an electronic differential analyzer. Thus, its objectives are highly specialized; it does not treat network analyzers, electro-mechanical differential analyzers, or any of the other analog computers. Furthermore, the author is interested in techniques of using analog computers, and only passingly in techniques of building such computers; in particular, he limits his discussion of engineering to the information which he feels is essential in proper application of the computer. In this application, however, he is insistent that the operator "feel" the way in which the machine actually functions analogously to the operation being simulated. Thus, he would like to have the operator understand why a differential equation describes a system being studied and why the same differential equation describes the operation of the analog computer used to find a solution; by bypassing the

differential equation, then, the operator understands how the analog computer simulates the system to be studied.

In order to get on with his task, the author limits himself bibliographically to those works which are of almost immediate importance. It is somewhat shocking to find a book on analog computation in whose index the name of Vanevar Bush appears only once. The Massachusetts Institute of Technology is likewise listed only once, and the name of S. H. Caldwell is not in the index. Thus, explicit reference to the great pre-war development of differential analyzer techniques at M. I. T. is almost completely lacking. However, the reviewer can see no real necessity for recalling old times in a strictly utilitarian textbook; the bibliographical omissions were mentioned above to illustrate the utilitarian character of the book and not to criticize the author's scholarship.

The one departure from the restrictions to electronic differential analyzers occurs in the last chapter when incremental digital computers are discussed (the so-called digital integrating differential analyzers). The thought here is that their use is more similar to that of the electronic differential analyzers than to that of the other digital machines.

The material in the text is presented in a lucid and elementary way. It is accompanied by problems for the student and numerous illustrations. Each chapter has a selected list of references, short but seemingly adequate for the material involved.

The reviewer feels that the book well attains its fundamental purpose of presenting the material that must be known to a person who intends to operate a standard electronic differential analyzer effectively.

The table of contents is: Chapter 1. Introduction; Chapter 2. The Linear Computer Components; Chapter 3. Time- and Amplitude-scale factors; Chapter 4. The Synthesis of Servomechanism Systems; Chapter 5. Multiplying and Resolving Servos; Chapter 6. Additional Computer Techniques; Chapter 7. The Representation of Nonlinear Phenomena; Chapter 8. Multipliers and Function Generators; Chapter 9. Miscellaneous Applications of the Electric Analog Computer; Chapter 10. Analog Computer Components and Computer Control; Chapter 11. The Checking of Computer Results; Chapter 12. Repetitive Analog Computers; Chapter 13. The Digital Integrating Differential Analyzer.

C. B. T.

46[X].—A. S. HOUSEHOLDER, Supplementary "Bibliography on Numerical Analysis," A. C. M. *Journal*, v. 3, 1956, p. 85–100.

Contains 321 titles supplementing those in the author's book, *Principles of Numerical Analysis*.

C. B. T.

47[X, Z].—GEORGE E. FORSYTHE, "Selected references on use of high-speed computers for scientific computation," *MTAC*, v. 10, 1956, p. 25–27.

See also the author's "A numerical analyst's fifteen-foot shelf," *MTAC*, v. 7, 1953, p. 221–228.

C. B. T.

48[Z].—R. K. RICHARDS, *Arithmetic Operations in Digital Computers*, D. Van Nostrand Co., Inc., New York, 1955, iv + 397 p., 23 cm. Price $7.50.

So far as this reviewer is aware, this important book contains the most complete existing description of the application of Boolean algebra to the logical design of digital computers. It is already a standard handbook of computer design used with profit by design engineers, students of the logic of computers, and students of the logic of such operations as microcoding.

The author prepared the contents as a set of notes to be used in connection with a course offered to engineers in the laboratories of the International Business Machines Corporation. As such it is strictly utilitarian. Its bibliography is incomplete. In the index one does not find the name of von Neumann, and the name of Claud Shannon occurs just once; the Shannon reference credits him with appearing to have pointed out the adaptability of Boolean algebra to the design of switching circuits. Similarly, the index lists thirteen references to IBM, but none to any of the groups which have formed Sperry Rand.

The reviewer finds no quarrel with this detachment and biased reference. However, the lack of complete references precludes use of the book as a reliable basis for estimating the complete state of the science at the time the book was written. The author had a goal which he met well. He wanted to describe design of computing instruments and he turned to the material at hand. Where it was convenient and where it served his purpose he took material from other sources, and to the extent that he had information readily available concerning the ultimate source he seemed to give it. Thus he gives no fewer than a dozen examples of binary adders in block diagram form (using Boolean elements) and discusses their relative merits. He goes into considerable detail concerning the difficulties of designing decimal components and again he illustrates his arguments with many block diagrams. In similar fashion he examines the other every-day problems of the designer of circuits for computers. He claims no new results and hence he feels little need for discussing relations between his results and those of other workers.

There is no discussion of the practical limitations of circuit design. Thus the deterioration of pulses on transmission lines and the effects of such deterioration on the logical functioning of the circuits (with the implications which can be drawn concerning the speed of computation which is safe) are not discussed in detail. Indeed, only the most idealized engineering realizations of the Boolean elements are discussed at all. These realizations are limited to vacuum tube circuits with no discussion of logical elements built of transistors and magnetic cores.

Similarly, the reduction of Boolean expressions to most economical form is discussed by implication only and no thorough analysis is attempted.

These omissions are not shortcomings, for the book is a thorough exposition by example of much of the formalism which is necessary for an effective application of knowledge in these omitted fields to the design of computers. Thus, there is no better place for an engineer or other person not already well informed concerning symbolic logic and its application to switching circuits to learn how to combine the components which may be developed from new electronic devices (such as transistors) into effective computing instruments. Similarly, there is no

other source which seems to give as easy access to the ideas involved in studies of fast and complex circuitry (such as [1], for example).

The author has included a chapter on computer organization and control and one programming. These are fairly superficial accounts suitable for project engineers but certainly not sufficiently complete to serve as serious expositions of these complicated subjects. Presumably these chapters were added to describe the general machine to the engineer working on its components. However, the subjects treated are not completely within the scope of Boolean algebra (as the author notes on p. 339) and hence not completely within the scope of this book.

In short, the author has prepared a well-directed set of notes for use as a practical handbook for anyone interested in the inside of an electronic computing instrument. The book contains no problems for solution by the reader, but otherwise it is entirely suitable for use as a textbook for a course covered by the material accorded attention.

The table of contents follows. Chapter 1. Symbolic Representation of Quantities, Chapter 2. Boolean Algebra Applied to Computer Components, Chapter 3. Switching Networks, Chapter 4. Binary Addition and Subtraction, Chapter 5. Binary Multiplication and Division, Chapter 6. Decimal Codes, Chapter 7. Counting, Binary and Decimal, Chapter 8. Decimal Addition and Subtraction, Chapter 9. Decimal Multiplication and Division, Chapter 10. Miscellaneous Operations, Chapter 11. Computer Organization and Control, Chapter 12. Programming, Bibliography, and Index.

C. B. T.

1. A. WEINBERGER & J. L. SMITH, "A One-Microsecond Adder Using One-Megacycle Circuitry," IRE Trans. on Electronic Computers, v. EC-5, 1956, p. 65–73. This article also appears under the title, "The Logical Design of a 1-Microsecond Parallel Adder using 1-Megacycle Circuitry," in Western Joint Computer Conference, *Proc.*, Feb. 7–9, 1956, San Francisco, California, sponsored by The Am. Inst. of Elec. Engineers, The Assn. for Computing Machinery, and the Inst. of Radio Engineers. Pub. by Am. Inst. of Elec. Engineers, New York, 1956, p. 103–108.

TABLE ERRATA

Reviews in this issue mention errata in the following works:

D. K. C. MACDONALD & LOIS T. TOWLE, "Integrals of interest in metallic conductivity," Review **37**, p. 38–39.

MICHIO TAKASHIMA, "Tables for testing randomness by means of lengths of runs," Review **27**, p. 33.

NOTES

Societa Italiana per il progresso delle scienze

The 46th congress of this Society met in Sicily, 15–21 September 1956. During the meeting the fiftieth anniversary of its foundation was celebrated. In the inaugural session, which was attended by Prime Minister Segni, Professor Mauro Picone paid an eloquent tribute to Vito Volterra, who was one of the original group which founded the SIPS. Professor Picone, who is the Director of the Istituto Nazionale del Applicazioni del Calcolo, organized the Mathematics Section of the Congress, the theme of which was the progress in mathematical analysis due to automation; the Secretary of the Mathematics Section was Dr.

D. Caligo. After an introductory address by Professor Picone, elucidating the theme, a series of talks were presented. The titles, or free translations of them, follow.

L. Couffignal (Paris), Mathematical science and computational technique.
M. V. Wilkes (Cambridge), Automatic digital computers and scientific research.
J. Todd (Washington), Computational problems concerning the Hilbert matrix.
A. van Wijngaarden (Amsterdam), The numerical summation of Stowly convergent series.
B. De Finetti (Rome), Applications involving logic and combinatorial analysis.
C. Böhm (Rome), On the automatic evaluation of the precision of computation.
F. H. Raymond (Courbevoie), Evolution of the concept of the structure of a digital universal computer.
L. Dadda (Milan), The Computation Laboratory of the Milan Technical University.
A. Ostrowski (Basel), On the solution of equations using Taylor's series.
O. Taussky (Washington), Computational problems involving matrices with integral elements.
M. J. de Schwarz (Rome), Automatic solution of a transcendental equation.
E. L. Aparo (Rome), On matrix equations.
J. Kuntzmann (Grenoble), Research on error evaluation of differential problems.
E. Biondi (Milan), Experiments on the numerical solution of various types of partial differential equations.
D. Caligo and D. Dainelli (Rome), Experiments on the numerical solution of ordinary differential equations on FINAC.
C. Strachey (London), Some matrix problems arising from the consideration of flutter in aircraft.

In addition to the above papers on numerical analysis, there was another, G. Grioli (Padua), Statics of continuous media, and a special address by G. Fichera, who has succeeded Professor Picone in the Chair of Mathematics at the University of Rome, on "Progress in some fields of mathematical analysis in the half-century 1906–1956."

The papers will be published in full in a special volume, so no account of them is necessary. The introductory remarks of the talk of Professor Kuntzmann are, however, well worth recalling. He noted that during recent years many powerful computational methods have appeared and that it is now appropriate that genuine mathematical effort should be applied to them, so as to determine the basic ideas and to enunciate and establish theorems.

Numerical analysts should find themselves quite at home scientifically in Italy. Professor Picone's influence predominates and he firmly believes in the union of the best in mathematics with the best available equipment: at present a Ferranti machine FINAC is at the Istituto in Rome. The productivity of his organization on the theoretical side has been phenomenal, and undoubtedly the acquisition of the new tool will result in distinguished contributions to true numerical analysis. There is another center of automatic computation in Milan.

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Sir Edmund Whittaker

1873–1956

Sir Edmund Taylor Whittaker, Kt., M.A., Sc.D. LL.D., F.R.S., Professor Emeritus in the University of Edinburgh, died in Edinburgh, Scotland, on March 24, 1956.

Born on October 24, 1873 in Lancashire, he received his education in Manchester and Cambridge, England. After lecturing for some years at Cambridge, he was appointed Royal Astronomer of Ireland and Professor of Astronomy in Trinity College, Dublin in 1905 and in 1912, Professor of Mathematics (i.e., head of the Mathematics Department) in the University of Edinburgh, a position which he held until his retirement in 1946. During Whittaker's tenure of the Chair, the Edinburgh Mathematics Department achieved well-merited international fame, and today former members of that department occupy important academic positions all over the world.

Whittaker's scientific activity extended over more than half a century and over many branches not only of mathematics but also of astronomy, physics, and philosophy. If he was a great and scholarly mathematician, he was equally great as a teacher and administrator; and those who had the good fortune of knowing him well admired him to the point of veneration. His gracious urbanity, his humane interest in people and affairs, his readiness to listen to and help people in trouble, his thorough knowledge of history and inexhaustible fund of anecdote, and his skill as a gardener are some of the many facets of a rich personality now no longer among us.

A biography and an analysis of Whittaker's scientific work will appear elsewhere. Here it will be sufficient to say a few words of Whittaker's work in those fields of particular interest to readers of *MTAC*.

Shortly after coming to Edinburgh, Whittaker established a Mathematical Laboratory, the first enterprise of this kind in Great Britain, and probably one of the first attempts to make numerical analysis an integral part of the university curriculum. It is probable that in starting this new venture, Whittaker was influenced by his work in astronomy, and by his friendship with the great actuaries of the period; and to the knowledge of the present writer, he regarded the introduction of the Mathematical Laboratory course as his most notable contribution to mathematical education. He and his associates, among whom the most outstanding was A. C. Aitken, researched on curve fitting, numerical solution of integral equations, and similar topics; and his thorough knowledge of the history and practice of numerical analysis found expression in his *Calculus of Observations* (written in collaboration with G. Robinson) which today, more than thirty years after its appearance, and after two revolutions in numerical computing, is still one of the most important works on the subject.

Much of Whittaker's work in analysis is distinguished by a trend towards practical applications. He was not satisfied with the mathematical investigation of a function unless means were found for its computation. The number of his pupils and associates in this field is legion; the one who developed the "practical" orientation to the greatest extent is E. L. Ince whose work on Mathieu and Lamé functions is a point in case. To this field belongs Whittaker's most widely known

work, *A Course of Modern Analysis*, which, especially since its second edition in 1915 (prepared in collaboration with G. N. Watson), influenced generation after generation of analysts.

In the course of a long and fruitful life Whittaker received many honours and distinctions; a bare enumeration of these would double the size of this brief notice. At the early age of thirty-one he was elected a Fellow of the Royal Society, and half a century later he received the Copley Medal of that Society. Among the numerous honours those probably giving him the most pleasure were his election as an Honorary Fellow of Trinity College, Cambridge, and his appointment to the Pontifical Academy of Sciences accompanied by a Papal Medal.

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CORRIGENDA

Review 97[K], *MTAC*, v. 10, 1956, p. 243,

for G. W. THOMPSON, *read* G. W. THOMSON.

The same misspelling occurs on page 267, line 5 (first column), and also on page 268, line 16 (second column), in the Index.